Barn-Raising an Endo-Pentakis-Icosi-Dodecahedron

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Abstract

The workshop is planned as the raising of an endo-pentakis-icosi-dodecahedron with a 1 meter edge length. This collective experience will give the participants new insights about polyhedra in general, and deltahedra in particular. The specific method of construction applied here, using kite technology and the snowflake layout allows for a perspective entirely different from that found in the construction of hand-held models or the observation of computer animations. In the present case, the participants will be able to pace the area of the flat shape and physically enter the space defined by the polyhedron.

Introduction

1.1 The Workshop. The event is set up so that the audience participates actively in the construction. First, the triangles are laid out by six groups of people in order to complete the net. Then the deltahedron is assembled collectively, under the direction of the artist.

The large scale of the event is designed expressly to give the participant a new point of view with regards to polyhedra. Instead of the "God's eye view" of hand held shapes, or the tunnel vision allowed by computer modelling, the barn-raising will give the opportunity to observe the deltahedron on a human scale, where the shape's structure is tangible in terms of the whole body, not limited to the finger tips and the eye. This should give rise to new intuitions about polyhedra.

1.2 Shape Definition. An endo-pentakis-icosi-dodecahedron is a complex deltahedron made of 80 equilateral triangles (figure 1c). It can be defined in two different ways. First, by taking a dodecahedron (figure 1a) and truncating it so that the pentagonal faces have rotated by 36 degrees and the vertices have become equilateral triangles (figure 1b). Then the pentagonal faces are "dimpled in" by replacing them with a concave 5-pyramid made of equilateral triangles. The other way to proceed is to start with an icosahedron (figure 1e) and subdivide each face into 4 equilateral triangles (figure 1d) and then "dimple in" each existing 5-vertex, using only the nearest 5 triangles.

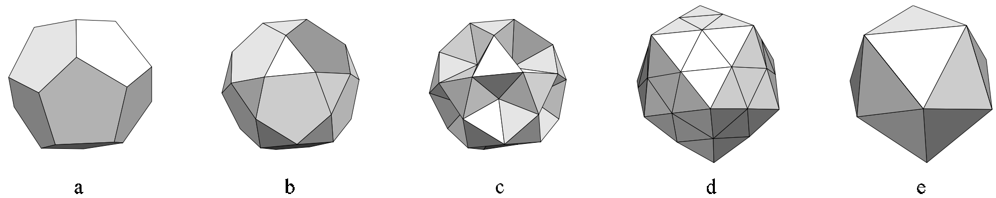

Figure 1: *From the dodecahedron or the icosahedron to the endo-pentakis-icosi-dodecahedron*

Background

2.1 Deltahedra. The endo-pentakis-icosi-dodecahedron is part of the class of polyhedra known as deltahedra [1]. These solids are defined as polyhedra whose faces are all equilateral triangles. Though there are only 8 convex deltahedra, there are infinitely many non-convex ones, including the present example. Other examples include the tetrahedron, octahedron, icosahedron, and the stella octangula (figure 2).

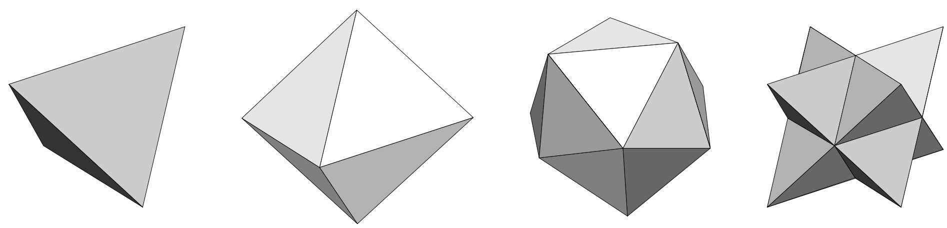

Figure 2: *Tetrahedron, octahedron, icosahedron and stella octangula*

2.2 The Process of Construction. This method, although different from the traditional method, is significant from the point of view of the experience it provides, and from the relationship between the net and the completed volume (figure 3).

From the point of view of the artist, the use of the snowflake-like shape emphasizes the fact that the structure of the two related shapes (the polyhedron and the net) are visible in either. This is of course due to the presence of deliberately placed overlapping elements (tabs), shown in grey in the figure. In technical terms, this shows a subtractive method, where excess material is removed, similarly to the chiselling away occurring in stone sculpture, as opposed to the additive method used in traditional nets where faces are added until the polyhedron is closed up (reminiscent of casting in bronze sculpture).

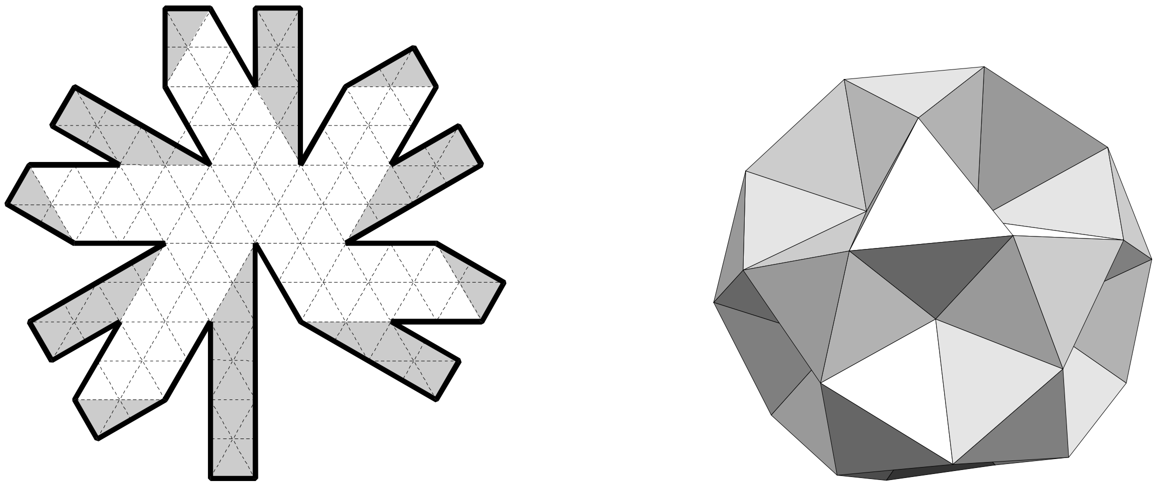

Figure 3: *Endo-pentakis-icosi-dodecahedron and its snowflake-net*

From the mathematician's point of view, the snowflake net emphasizes the application of the Euler characteristic (F+E-V=2)[2], as well as the fact that the total angle deficit of closed shapes is a constant 4Π, the Gauss-Bonnet Theorem for polyhedra [3].

2.3 Applications in Mathematics Education. The many laws and theorems of polyhedral geometry this method illustrates as well as the interesting perspective of observation it provides makes this workshop an excellent tool for mathematics education at the middle and high school level. The project is in fact being applied as part of the Rice University School Mathematics project, where the deltahedron is being raised as part of an art-week event by a group of middle school students and their mathematics teacher in Houston, Texas. The elements of the construction are designed in a modular fashion so that they can be used to build different deltahedra. In fact, an icosahedron, an octahedron, a stella octangula, an endo-tetrakis-hexa-octahedron (dimpled truncated cube) and a few others have already been built, using the same material.

Practice

3.1 The Process of Construction. Though the workshop accompanying this paper is complete in itself, the method of construction of the deltahedron includes a few additional steps necessary to define the net that determines the deltahedron. We have included these for teachers who want to reproduce this workshop with plain paper models. The following method can be modified for the construction of other deltahedra.

3.2 Constructing the Net.

1. Start with a section of triangular grid bounded by a regular hexagon of side length 6 . Number the vertices of the hexagon clockwise from 1 to 6, starting with the upper left hand corner. The center will be referred to as 0 (figure 4).

2. Mark off a wedge of 30^{o} from the center point 0 towards the right half of the edge 5-4. This wedge (A) will be cut out later (figure 5).

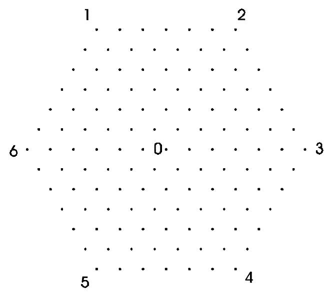

Figure 4: *Basic grid*

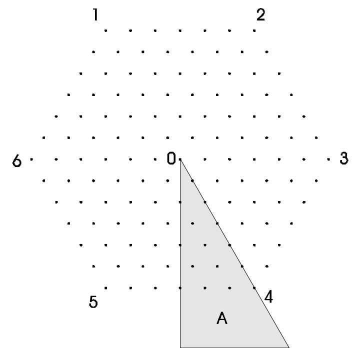

Figure 5: *Incision A*

3. Along the line 0-4, travel down 2 units, then turn counter-clockwise by 30^o and draw a line towards the edge 4-3. This line, together with the line 0-4, marks off a new wedge (B). Repeat the procedure starting at 0 and following the lines 0-3, 0-2, 0-1, 0-6 and 0-5 (figure 6, wedges C, D, E, F and G).

4. Starting again at 0, travel 4 units towards vertex 3, mark off a 30^o wedge along 0-3 between 3 and repeat along 0-2, 0-1, 0-6 and 0-5 (figure 7, wedges H, I, J, K and L).

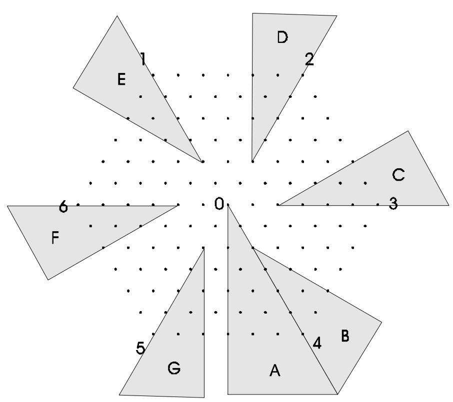

Figure 6: *Incisions M, N, P, Q and B, C, D, E, F and G*

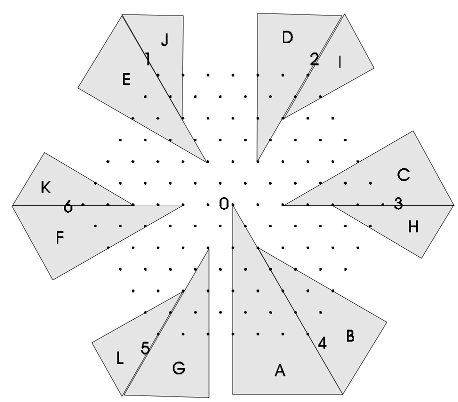

Figure 7: *Incisions H, I, J, K and L*

5. Finally, starting at 0, travel two units towards 3, then two units in the 0-4 direction. This should bring you to a point that is located at "2 rhombi" of 0 towards the edge 3-4. From there, draw a line towards the 3-4 edge, perpendicularly to the edge, and one in the 0-3 direction. This marks off a 30^o wedge. Repeat the procedure in the areas 0-3-2, 0-2-1, 0-1-6, and 0-6-5 (figure 8, wedges M, N, P, Q and R). This part is necessary because of the overlap from step 3.

6. Cut out the hexagon, then cut off all the marked off areas. You should now be left with a rough snowflake-like shape with a piece missing (figure 9).

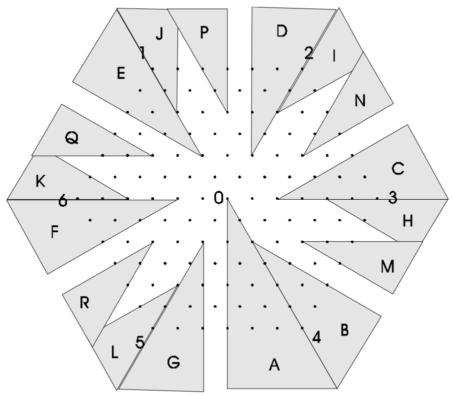

Figure 8: *Incisions M, N, P, Q and R*

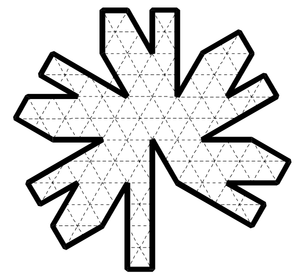

Figure 9: *The completed snowflake*

3.3 Assembling the Deltahedron. Once the net has been defined, there remains only to fold and assemble the deltahedron. To do this, the paper must first be folded along all the lines of the initial grid, in all three directions. Experience has shown that in the case of hand-held models, it is easier to use adhesive putty between the overlapping layers to assemble the shape. There now remains only to overlap the appropriate areas to finish assembling the deltahedron.

1. Overlap the two branches adjacent to the first incision (grey areas in figure 10) so that thinner branch is under the other, and so that the 30-60-90 triangles are hidden. It is important that the folds are made in such a way that the paper resembles a mountain with a 5-triangle crater at the top (the first dimple).

2. Apply the same process to the incisions made in step 3 of the previous section, again creating 5-triangle dimples around the first one, overlapping the branches so that the 30-60-90 triangles are hidden underneath (grey areas in figure 11). There are five of these.

3. Repeat the preceding step with the third row of incisions (the grey areas in figure 12). There are five of these. In this final step, it is important to fold in the outermost triangles since they collectively constitute the last dimple.

4. Once the last dimple is tacked using the putty, you are done!

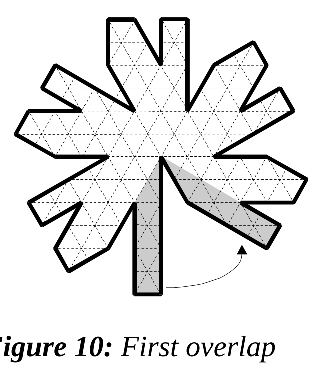

Figure 10: *First overlap*

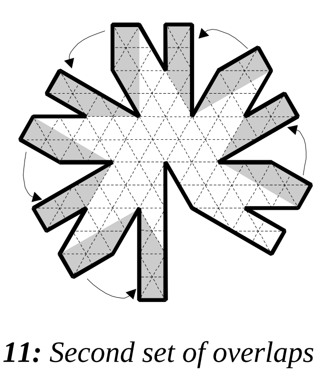

Figure 11: *Second set of overlaps*

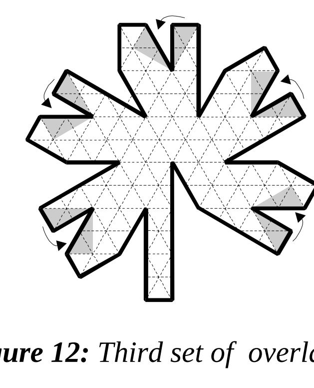

Figure 12: *Third set of overlaps*

Conclusion

4.1 Further research. This project as a whole is of course far from being concluded. The potential for further exploration is practically limitless and the knowledge that can be found through it has not yet run dry. The ever-present coloration problem, and its related topic, the symmetry groups can still be explored, particularly in light of the visible relationship between the deltahedron and the corresponding net. Research can also be made about the different "snowflakes" that determine each deltahedron. There can of course be more than one different snowflake per deltahedron depending if the layout is centered around a face, an edge or a vertex. Further experimentation can be made regarding the snowflake net of deltahedra containing vertices of degree higher than six such as the stella octangula. Finally, how would the net look like for deltahedra of genus higher that 0? These would include tori made of equilateral triangles and such. These questions have not yet been answered, but the results achieved thus far look promising.

Acknowledgements

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References

[1] Cundy, H.M. and Rollet, A.P., *Mathematical Models*, Revised Edition, Oxford at the Clarendon Press, 1961. Pages 78 and 142.

[2] Ibid. Page 89.

[3] Alexandrov, A.D. and Zagaller V.A., *Intrinsic Geometry of Surfaces*, Translations of Mathematical Monographs vol. 15, American Mathematical Society, Providence, Rhodes Island, 1967. Page 8 and Do Carmo, M.P., Differential Geometry of curves and Surfaces, Prentice-Hall Inc., Englewood-Cliffs, New Jersey, 1976. Page 274.