

Investigating ‘Epistemologically Correct’ Experiences of Mathematical Learning Expériences d’apprentissage mathématique « épistémologiquement correctes » : une investigation

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Introduction

In her 2004 report on mathematicians as learners, Burton cited Wenger:

Educational processes based ... on actual participation are effective in fostering learning not just because they are better pedagogical ideas, but more fundamentally because they are ‘epistemologically correct’, so to speak. There is a match between knowing and learning, between the nature of competence and the process by which it is acquired, shared, and extended (Wenger, 1998).

The project that this report presents focuses on the application of this perspective to a mathematics course for elementary student teachers. To this end, a framework describing ‘epistemologically correct’ mathematics learning experiences was outlined. Based on this framework, a description of the type of tasks that could be considered to lead to such experiences was developed, and an undergraduate course that encapsulates these intentions was designed and implemented. This implementation was then evaluated in terms of the intervention’s success at providing ‘epistemologically correct’ experiences (as defined) of mathematics and in terms of changes in the affective responses of the participants. The report focuses on the first two points: the development of a framework for developing teaching approaches that provide the appropriate experiences, and the resulting teaching approach. The thesis itself also included the evaluation of the approach in terms of its intention, and of its influence on the participants’ affect.

Integrating a Historical Perspective on Epistemology

In order to define what could be considered ‘epistemologically correct’ experiences of mathematics learning, I investigated the literature on the philosophy of mathematics, with a particular look at the changing perspectives throughout history. This survey revealed a movement from what was mostly empirical observation, to views that were more and more rationalist, with a peak in the early twentieth century, at which time the philosophical intention was to completely explain all of known mathematics using only a system of pre-established rules (logic and set theory), thereby eliminating the empirical ontology altogether. This intention is well known to have failed due to the extreme nature of the underlying view. Lakatos (1976) provided a more balanced, integrated view of the development of mathematical ideas, suggesting that the perspectives are not incompatible, and can be integrated.

Though a widely held contemporary perspective is that mathematics is socially constructed, in practical terms, in the classroom, empirical observation and the idea of Universal Truth are very much alive. To reconcile the various views, as Lakatos has done, it is possible to develop a pluralistic, non-homogenous framework of the epistemology of mathematics. This framework integrates empirical observation, logical reasoning and social norms and consists of four types of mathematical notions¹:

- Observational notions, which are the result of basic empirical observation, without explanatory content. The epistemological source of these notions lies in personal sensory experience with a phenomenon.
- Conventional notions, which are not the result of empirical observation or logical derivations from more basic or fundamental notions. They have been chosen by the experts or imposed by simple enculturation (Pimm, 1995) as convenient for the task, are accepted socially, remain unquestioned, and are treated as monolithic.
- Applicational notions, which are the product or application of some form of mathematical reasoning upon the previous two categories, and can therefore be explained and traced back to this reasoning.
- Theoretical notions, which make possible the reasoning that itself produces the applicational notions and promotes their adaptability.

The four types of mathematical notions are all part of the wider body of work that constitutes what is today accepted as mathematics. The typology of a specific notion, however, is not necessarily fixed. It is possible to treat a notion as conventional, even though it was derived empirically or logically. This is often done when the specific notion is left unquestioned and simply *used* to solve a problem, even if the user knows it as applicational or even theoretical. Any mathematical notion, therefore, can be ‘treated as’ conventional, or observational, even though it is also potentially applicational, for the purpose of the task at hand. For this reason, a framework describing the way of knowing these notions must be superposed onto this typology.

The simplest types of notions, conventional and observational, tend to be perceived as monolithic and left unquestioned. If known, they can be replicated when needed. Familiarity with them can be expressed as ‘knowing-what’ or ‘knowing-how’, in the case of a process or algorithm.

Applicational notions can be known that way too, but familiarity can also be expressed as ‘knowing-why’ when the underlying reasoning is familiar enough to bear examination.

Finally, theoretical notions can be treated like conventions, or reasons for other notions, but they can also be expressed using ‘knowing-when’, that is, as framing domains of application, or ways of reasoning about, etc. An individual ‘knows when’ to apply the notion.

The table below shows a summary of the three ways of knowing mathematical notions, and several of their properties, including the ‘thinking register’.

¹ I interpret mathematical knowledge as being about mathematical objects and of relationships between them. In addition, these relationships can themselves be regarded as objects of higher-level relationships, as in the example of mathematical logic, which uses relational statements as objects. At any given moment in a mathematical situation, therefore, mathematical statements are treated either as statements of relationship, or the object of relationships with one, several or a whole class of other mathematical objects. For this reason, and to lighten the flow of the text, I use the term (mathematical) notions to include both (mathematical) objects and (mathematical) relationships, unless the distinction is significant (Knoll, 2007).

Table 1: Levels of Knowing Mathematical Notions

Level	Knowing-what/how	Knowing-why	Knowing-when
Register	Low level	Mid-level	High level
Manifestation	Recalling a fact or performing a process	Monitoring a process	Abstracting from a process
Engagement	Passive	Active	Critico-Creative
Properties	Replicative	Transferable	Constructive
Perception	Determined by 'external Authority To be memorised	Instance of reasoning behind a notion	Rigour in mathematical structure. Can be used to (re-)construct

A mathematical learning experience, to be 'epistemologically correct', must incorporate all these levels. Tasks that promote this experience, therefore, need to present the engaged participant with an opportunity to use and develop knowledge about notions at all the levels. I define such tasks in the next section, and call them 'Mathematical Enquiry'.

Classifying Mathematical Tasks

In order to examine a mathematical task in terms of the levels of the preceding typology, it is necessary to evaluate its potential for *knowing-when* as well as *-why* and *-what/how*. To facilitate this evaluation, I propose a 2-dimensional classification system based largely on the literature on problem solving.

Routine versus Non-Routine Problems

Firstly, a task can be classified according to the level of engagement that is required to perform it successfully. This classification has been expressed as the difference between a problem, defined as a "*situation where the [participant] cannot at once decide what rule to apply or how it applies*", and an exercise defined as "*a situation in which this is at once obvious*" (Passmore, 1967; see also Zeitz, 1990). This distinction is similar to that made by Polya (1957) between routine and non-routine problems. An important aspect of this classification is that it is not simply a condition of the task itself, but of the participant's familiarity with it. A specific situation can evoke a routine exercise for a more knowledgeable individual, or great puzzlement for a less knowing one. This focus on the participants' readiness to perform the task is well developed in Goldin's (1982) framework, which focuses more on this condition and develops a 5-fold distinction. In his framework, a task can belong to any one of the following categories:

1. The subject 'knows the answer' or is already at the goal when the task is posed.
2. The subject does not 'know the answer,' but 'possesses a correct procedure' for arriving at it
3. Same as 2, but the subject is unable to describe the procedure in advance of carrying it out.
4. Same as 3, but the subject 'does not know for sure' (cannot state with certainty) that he or she possesses the procedure until after the problem has been attempted.
5. The subject does not possess a procedure for arriving at the answer (pp. 95-96)

In this categorization, the participant's possible relationship with the task is much more differentiated than in the previous versions, and the obstacle can be of various natures. At Goldin's level 2, for example, although the participant knows what procedure to carry out, suggesting that he only needs 'knowing-what/how', s/he cannot explain it "in advance of carrying it out". 'Knowing-when' is not required. In 1992, Mason advanced a different definition:

I take the word problem to refer to a person's state of being in question, and problem solving to refer to seeking to resolve or reformulate unstructured questions for which no specific technique comes readily to mind (p. 17, footnote).

The key to Mason's statement is the student's state of being when performing a task, her/his engagement with the problem. This condition of problem solving is neither a feature of the problem, nor of the solver, at the start of the solving process: it is a property of the relationship between the two. It thereby connects, through the level of engagement, to the level of knowing required, as shown in the table.

In summary, mathematical tasks can be classified according to how routine they are for a specific individual. Mathematical enquiry, if it is to require all the levels of knowing described above, needs to be as non-routine a task as possible, so as not to engage solely the lower levels.

Problems Requiring Mathematics to be Solved, versus Problems of a Mathematical Nature

A second distinction can be made, between problems that require the application of mathematics, and problems that are mathematical by their very nature, that is, those which, if solved, produce new mathematics, at least for the solver. This distinction is akin to that made by Polya (1957), between "problems to find" and "problems to prove". According to his description:

The aim of a "problem to find" is to find a certain object, the unknown of the problem. [...] We may seek all sorts of unknowns; we may try to find, to obtain, to acquire, to produce, or to construct all imaginable kinds of objects. [...] The principal parts of a "problem to find" are the unknown, the data and the condition (p. 154-55).

This corresponds to the category of problems "that require mathematics to be solved". In contrast, the resolution of problems that are mathematical in nature elicits, on the part of the solver, the development and creation of a mathematical system. The distinction could be considered one of focus, between the search for a solution to a concrete, specific application, or in general to a class of situations, with the corollary that the boundaries of the domain of applicability need to be defined. If the focus is on a specific application, the solution is the goal-state and work can stop when this is found. If generalisability is sought, the problem can be pursued further by examining similar cases, classifying them, working on defining boundaries, special cases, etc. This suggests the use of knowing-when. Problems of a Mathematical Nature are therefore good candidates for mathematical enquiry.

Defining the Criteria for Mathematical Enquiry

The classifications described above form the basis for the conditions for mathematical enquiry to take place. They do not, however, pose specific criteria for the design of such tasks. Such a framework can, however, be reified into a series of criteria, as follows. In 2003, Grenier and Payan's developed a framework designed to determine the conditions for engagement, on the part of school children, in 'professional research situations'. For them, it was essential that:

En situation de recherche, le chercheur peut, et doit, pour faire évoluer sa question, choisir lui-même le cadre de résolution, modifier les règles ou en changer,

s'autoriser à redéfinir les objets ou à modifier la question posée. Il peut momentanément s'attaquer à une autre question si cela lui semble nécessaire.²

Researchers have ownership of their enquiries in a way that allows them to redefine, modify or even temporarily abandon them in favour of another. To replicate this kind of context requires the development of criteria for the design of a teaching approach that is intended to provide experience with mathematical enquiry. These criteria are delineated below.

Criterion 1: A Novel Starting Point

Une SRC s'inscrit dans une problématique de recherche professionnelle. Elle doit être proche de questions non résolues. Nous faisons l'hypothèse que cette proximité à des questions non résolues - non seulement pour les élèves, pour l'ensemble de la classe, mais aussi pour l'enseignant, les chercheurs - va être déterminante pour le rapport que vont avoir les élèves avec la situation.³

A research situation, to be acceptable according to their criteria, needs to be unsolved as far as the whole community of mathematics is concerned, in order to ensure full, 'professional' engagement. This is not always practical, and in any case, as Mason (1978) explains:

The question by itself cannot replace the process leading to its articulation, so the student is not in the same state as the originator. (p. 45)

In this project, rather than making use of unanswered mathematical problems, the participants developed their own starting point. In the larger sense, a task can be called one of mathematical enquiry if the starting point is new to the (local) participant(s). If they create it, it is new to all the participants, because the originator holds the meaning of the problem.

Criterion 2: An Open-Ended Process

Plusieurs stratégies d'avancée dans la recherche et plusieurs développements sont possibles, aussi bien du point de vue de l'activité (construction, preuve, calcul) que du point de vue des notions mathématiques.⁴

According to Grenier and Payan, a mathematical task, to emulate professional activities, must not prescribe a specific method of resolution. The process must remain the choice of the student-participant. This is also true of mathematical enquiry.

Criterion 3: An Open-Ended Goal-State

Une question résolue renvoie très souvent une nouvelle question. La situation n'a pas de « fin ». Il n'y a que des critères de fin locaux.⁵

² In a research situation, for the problem to evolve, the researcher can and should determine the domain of applicability of his questions, modify or replace the rules under which s/he operates, allow her/himself to redefine the objects of the problem or indeed the problem itself, focus temporarily on a different question if it seems necessary. (my translation)

³ An RSC [Research Situation for the Classroom] is framed by a professional research question. It must be connected to problems which are unsolved in the canon. We make the hypothesis that the fact that the problem is unsolved, not only for the pupils, but for the instructors and for the participating professionals, is key to the rapport which the pupils will develop with the situation. (ibid)

⁴ Several investigation approaches and developments are possible, both from the point of view of the activity (construction, proof, calculation), and from the point of view of the mathematical knowledge. (ibid)

⁵ An answered question often leads to a new question. The situation has no 'goal-state'. There are only criteria of local resolution. (ibid)

As discussed previously in the description of problems that are mathematical in nature, a task, to be considered mathematical enquiry, must have an open-ended goal state: any solution will lead to new questions, of generalisability, of domains of applicability, of special cases...

Criterion 4: An Atmosphere of Security

La question initiale est facile d'accès : la question est « facile » à comprendre. Pour que la question soit facilement identifiable par l'élève, le problème doit se situer hors des mathématiques formalisées et c'est la situation elle-même qui doit « amener » l'élève à l'intérieur des mathématiques. Des stratégies initiales existent, sans que soient indispensables des prérequis spécifiques. De préférence, les connaissances scolaires nécessaires sont les plus élémentaires et les plus réduites possibles.⁶

The uncertain nature of mathematical enquiry needs to be mitigated, for participants to feel safe in engaging actively, by the social context of the experience. In particular, in the classroom context, a participant will engage in an activity if it is deemed feasible, under the didactic contract and based on the knowledge available. This is an essential criterion for the implementation of mathematical enquiry in the classroom as it is a condition for engagement.

Criterion 5: The Experience of Mathematical Enquiry

In addition to the above-mentioned criteria, mathematical enquiry, to be authentic, needs to be experienced in a way that is analogous to the experience of professional mathematicians. Descriptions of such experiences often are expressed in terms of stages, as in Hadamard's scheme (see Liljedahl, 2004), which includes preparation (initiation), incubation, illumination and verification, or Mason's (1992) energy states. A common feature of these descriptions seems to be the importance of giving the experience time to unfold. Burton (2004), for example, argues:

The strategy of a student [...] of having time and space to retreat, reflect, research, is not only appropriate to the unsolved problems of research mathematicians. Students undertaking a mathematical challenge also need to have room to manoeuvre, to work together, to consult people or books, to think.

Time, therefore, needs to be given for risky avenues to be investigated, questions to be reformulated, ideas to be incubated and results to be examined and re-examined.

Additional constraints were derived, in this project, from the research methodology and the context within which the study took place. The data collection consisted of both point-in-time (questionnaires, recorded class discussion) and continuous events (journals), promoting participants' reflections on their experiences both before and during the intervention, thereby transforming those experiences. The programmatic context of the course also influenced both its mathematics curriculum and the type of participants.

Chronology of the Implementation

The teaching approach resulting from this development was implemented in the context of an undergraduate mathematics course taken by students registered in an integrated bachelor of education programme in a Midwestern state university. The mathematics course forms one of several that are required of elementary student teachers, and is traditionally given by the

⁶The initial question is easily accessible: the question is "easy" to understand. For the question to be easy to identify, the problem must be situated outside of formalised mathematics, and must pull in the pupil. Initial strategies exist, without requiring specific pre-requisite knowledge. Preferably, this required knowledge is made minimal. (ibid)

mathematics department. The participants consisted of 33 women and 4 men, ranging from less than 21 years old (16) to over 25 (3). 24 were registered in the integrated education programme, 4 in the Masters' for teacher certification, and 4 as undecided. 4 were in their second year of undergraduate studies, 15 in their third and 12 in their last. Classes met twice a week, and were broken down into five distinct phases.

Phase 0 consisted of the first class and focused mainly on the collection of pre-intervention data gathering, using a questionnaire and recorded whole-class discussion, and the students' introduction to the course syllabus. During this class, expectations were disclosed, by explaining that the student would engage in open-ended mathematical investigation. Journals were also distributed in the second class.

Phase 1, which consisted of 10 classes, saw the participants engage in 'mini-projects', that is, small-scale tasks designed to model the type of activities that they would be required to engage in during the main project. These projects consisted of 'research situations', not 'well-formed questions', which, with some guidance, got the participants used to the pace. This phase was considered to provide the participants with a 'ramping up' of engagement. A typical instruction during this phase was as follows:

Once you have got a pattern, check [that] it works for different examples, then ask [yourself] why it may always be true. Do the simple cases very thoroughly.
(Blackboard)

Phase 2 lasted for 7 classes and consisted entirely of student-led activities, based on student-identified starting points, development and goal-states. Interactions with the instructor-team were kept very hands-off and consisted largely of responses to questions with questions: the enquiry was the students', and so the answers should be theirs. An example of such interaction, reported by the main instructor was as follows:

Rob [pseudonym] asked about his 'number of manipulations'. 'Is this one or two?' he asked. I [wanted to ask] him to define it his way, but he completed the sentence [for me] (Dr Zachary, personal journal, p. 38).

Many of the interactions between the instructor team and the students demonstrated willingness on the latter's part to engage critico-creatively. The research reports reflected this.

Phase 3, which also consisted of 7 classes, brought the class back to a more regular way of working. This phase was mainly implemented in order to 'catch up with the curriculum' so that the required content was addressed.

Finally, Phase 4 consisted in summative assessments and closing data collection. The students presented orally the results of their enquiries, an exam was administered that covered the content of Phase 3, the journals were collected one last time, and the participants responded to the post-intervention questionnaire. This phase lasted 4 classes, including the exam. The analysis of the journals revealed a strong engagement, on the part of the majority of participants, in 'mathematical enquiry'. The thesis elaborates on these results and on the affective responses.

Discussion

The project as a whole consisted not only of the development of the framework and derived teaching approach described here, but of an analysis of the authenticity of the experience in terms of the exemplar that is 'professional practice', and of the participants' affective responses and changes therein. Briefly, the experience seemed to be authentic for a majority of the participants a great deal of the time, and the responses, as predicted by the literature, showed significant changes in the participants' beliefs about mathematics rather than in their attitudes, known to be more stable.

One of the limitations of this teaching approach that stood out, as much from the practical considerations at hand as through the participants' own responses, was the scale of the time investment required. This limitation is not easily done away with, however; it

... is so fundamental to the practice that it is constitutive of one of the five design criteria of the approach. As such, it changes from a characteristic of the practice to a condition for its authenticity, and therefore the question becomes not one of adjustment, but of the worthwhile nature of the practice as a whole (Knoll, 2007).

Another issue, directly linked to criterion 1 regarding the participants' ownership of the starting point, arose in the case of students who, within the parameters of the intervention, could elect to opt out of the experience by choosing a mathematical topic and task with which they were so familiar and thereby reduced the practice to an exercise. This appeared to happen only with one student. Other limitations of the practice include the difficulty in assessing activities whose results are uncertain.

In the case in point, the practice was applied to the preparation of future teachers. The claim is that for teachers to be able to speak with authority about mathematics as a discipline, it is important for them to have experienced it in all its aspects, including its creation. The questions remain whether this practice promotes such authority and whether the practice is applicable to the teaching of mathematics in schools itself.

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