Preliminary Field Explorations in K-6 Math-Ed: the Giant Triangles as Classroom Manipulatives[1]

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Abstract

The present paper reports on children's investigations using the giant equilateral triangles from the Geraldine Project[2]. It took place at the De Zavala Elementary School as the initial stage of a project in mathematics education. The triangles are a part of a modular construction kit made using kite technology. Their size, sturdiness and light weight make them ideal for in-class activities with children of all ages and stages of development.

The school is located in a low socio-economic hispanic neighbourhood consisting of blue-collar families living in apartments and rental houses as well as small businesses and industries. Most of the students at the school are recent immigrants from Mexico or Central America or first generation born to immigrant families. Their parents have little or no education and are forced to work on jobs that entail long hours, frequently into the evening or night. This situation makes it difficult for parents to provide their children with appropriate support as students.

At this stage, the structure of the activities that make up lessons emerged from the response of the children as the activities were tried. This approach, despite its unplanned nature, allowed for the introduction of much mathematical content, and the attention of the children was relatively easy to catch and hold. The activities successfully combined the play aspect of the giant triangles with the mathematical concept explorations that the instructors overlayed. In some cases the children were allowed to build their own shapes, which were then examined with them. The outcome of these trial activities was then used as a basis for lesson planning in later stages of the pilot project.

Introduction

1.2 Background. The giant triangles used in the activities described in this paper have been used in more structured lesson plans and activities that were developed for middle and high school in 1999-2000[3]. Topics have included space-fillers, pyramids, Platonic solids, semi-regular deltahedra[4], tunnels and symmetries. Working with these large brightly colored triangles has proven to be a great motivator, giving the children the enthusiasm necessary to rekindle their natural curiosity in mathematics as well as practiceteamwork and creative problem-solving.

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[1] This project was partially funded by the Shell Oil Company.

[2] See Morgan S. and Knoll, E., *Polyhedra, Learning by Building: Design and Use of a Math-Ed. Tool* Bridges Conference Proceedings, 2000.

[3] ibid

[4] A deltahedron is defined as a polyhedron whose faces are all same-sized equilateral triangles. See http://mathworld.wolfram.com/Deltahedron.html for more information.

1.3 Teaching Method. In the activities described here we also try to emphasize and exploit teamwork and creative problem-solving. The general format of the lessons involved up to three activities where the two instructors or the children built shapes or made patterns, followed by a questions and discussion period. This allowed the children to develop observation skills, vocabulary or concepts. The outcome of each activity determined the nature of the follow-up activity in each lesson.

Activities

2.1 The activities. The activities are described according to grade level and range from kindergarten to grade 6. They can be adapted to the level of the specific group of students and cover concepts and skills such as counting, symmetries and transformations.

2.2 Kindergarten. Note: in these activities, the instructors mostly took care of tying the triangles together, then asked the children questions about the shapes.

2.2.1 Investigation of a single equilateral triangle

- How many sides does it have? How many corners? These simple questions are designed to get the children started with the careful observation of the subject. The terms edge and vertex can be introduced with old enough children.
- Which is the longest side? This is of course a trick question, since the triangle is equilateral, but asked in this way, the children are compelled to compare all three sides in pairs. Because of foreshortening, the children will often find one side to be longer or shorter. If that is the case, the instructors can place another triangle as reference on the floor and compare all three sides of the contested triangle against it to verify (figure 1). This activity introduces the concept of measurement using another object as a reference, in other words the transitivity of the concept of same.

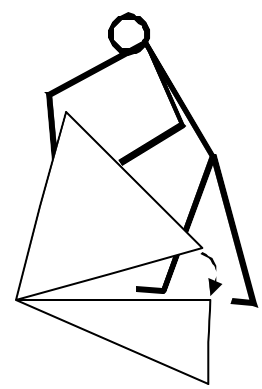

Figure 1: *Comparing to a reference triangle*

2.2.2 Assembling two-dimensional shapes.

- **Joining two triangles.** The children like to call this shape a 'diamond'. Depending on the situation, the instructors can introduce the term 'rhombus'.
- **Adding more triangles to the shape.** The children can see the shapes that are being built and like to compare them to things they recognize (see for example the cat in figure 5).
- **Playing the corner-sides game.** The instructors ask the children for each of the shapes how many sides (edges) and corners (vertices) there are. An observation can be made, that the number of sides and the number of edges are always equal for a given shape. To prove the validity of the statement, a game can be played. One child (or the instructor) moves around the perimeter of the shape and calls out each corner and side in turn. A second child keep tabs (with the fingers of their hand or marks on the blackboard) of the number of times the word side is called out and a third one does the same for the number of corners. After this is done for several shapes, the children are asked if they see a pattern. They will notice that for each shape the number of sides and corners are the same, and that the caller alternates calling out one, then the other. This can be considered an explanation as to why the two numbers are always the same. Note, shapes such as

the trapezoid (see figure 2), made of three triangles, should be avoided at this grade level as there is an ambiguity on the long edge of the trapezoid: is this one side, or two sides with a ‘flat corner’ in between?

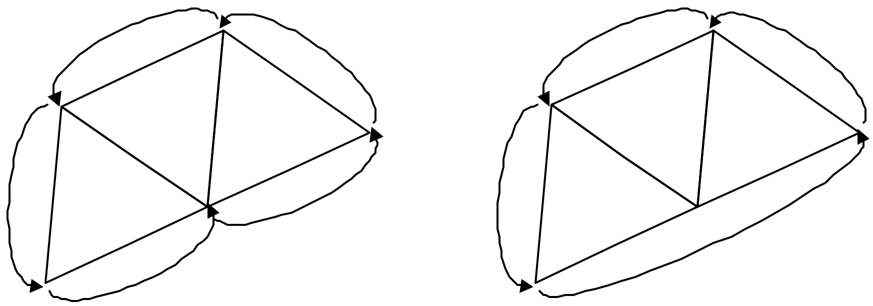

Figure 2: *four edges and four vertices, or five and five?*

2.2.3 Assembling three-dimensional shapes.

- **Examining the closed hexagon.** Can this be transformed into a pyramid? The instructors ask the children for suggestions how this could be accomplished. Suggestions may include lifting the hexagon up from the center (which of course doesn’t work because it will collapse again when it is released), folding over and tucking under two sides (see figure 3), and finally removing a triangle.

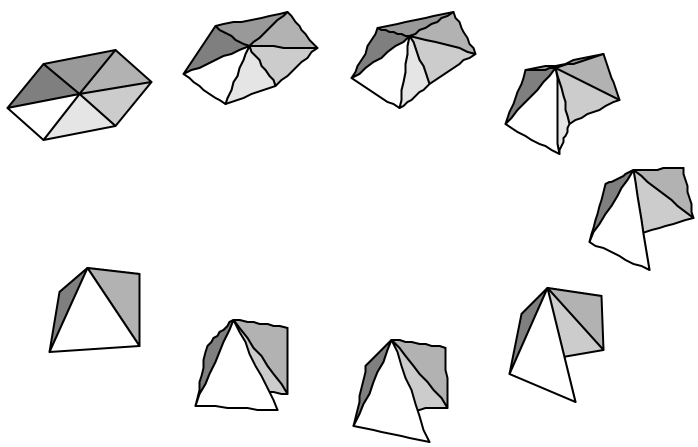

Figure 3: *Folding and tucking two triangles*

The clearest solution consists of removing one, two or three triangles and retying the remaining ones.

- **Building the pyramids.** After the instructors build the three-, four-, and five-triangle pyramids (in addition to the flat hexagon), the various shapes are compared. Properties to be discussed at this stage include height, pointiness (sharpness of the central vertex), size of the base shape, etc... Optional activity: Using masking tape (or some kind of marker), the base shapes of the pyramids can be traced on the floor and these shapes can be investigated on their own.

- **Building a shape that can be investigated from the inside.** Using the existing pentagonal pyramid, the instructors construct a partial icosahedron by completing the vertices presently lying on the floor (figure 4). The children can then file into the shape alone or in pairs and make observations. They can be asked whether they see “stars”, the 5-vertices.

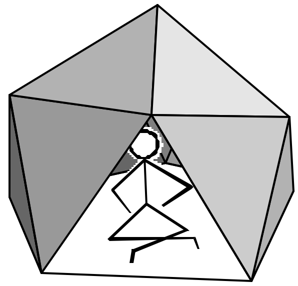

Figure 4: *The partial icosahedron*

- **Recording the experience.** At the end, the children can all make a drawing of what they experience. When tried in class, the drawings showed many different aspects of the experience, from an outside view to a connected set of lines representing the configuration of the surface itself. The instructors found the drawing activity itself to be very effective at kindergarten level, compelling the children to observe attentively what they see.

2.3 Grades 1-3. This exploration is the most open-ended: the children are divided into groups of four, given a dozen triangles, and asked to build shapes. Following this, the instructors assemble the whole class in front of each shape in turn and dialogue with the children about the shapes and their properties. In the lesson presented here, only two-dimensional shapes are discussed.

- **Investigating the shape as a whole.** At this stage, the children love to compare their work to physical objects. They will see as many different things in their work as one can see looking at clouds. Their imagination is titillated, and they show the instructors the cats, rockets, pacman, and Stars of David that they have made.
- **Finding simple shapes in the configurations.** At this stage, some of the children abstract parts of the configuration and point out smaller shapes; they see diamonds (rhombi) or edge-length-two triangles (figure 5).

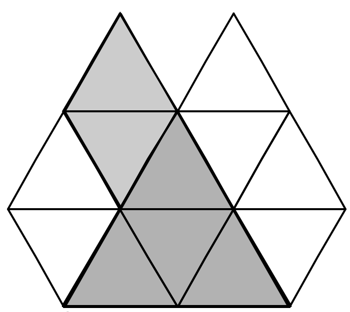

Figure 5: *The cat and two of its parts: a diamond and an edge-length-two triangle*

- **Counting triangles.** Once all the children have seen the partial shapes, they can count them. The potential difficulty in this activity comes from the need to count each triangle once and only once, even though they sometimes overlap. In the configuration of figure 6, for example, there are seventeen triangles: twelve small ones, four with an edge length of 2 and one large one with an edge length of 3.

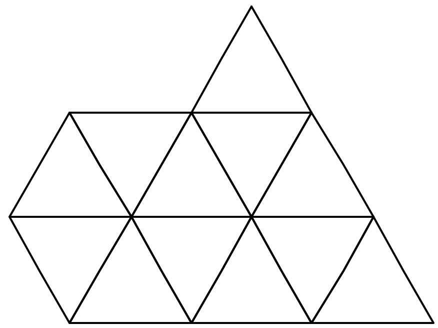

Figure 6: *Configuration containing a total of seventeen triangles*

When this part of the lesson was done at the elementary school, one of the children posed a noteworthy challenge: in the Star of David (figure 7), she counted *twenty-four* triangles. After asking her to show them, one of the instructors took her aside to get to the heart of the problem. In the Star of David, the little girl counted 12 small triangles, 6 medium-sized and 6 large ones. She counted each large one three times, clearly demonstrating with her open arms each angle of either triangle. When the instructor showed her an edge-length-two triangle as a whole configuration, she counted 7 triangles: 4 small ones, and again, 3 medium ones, counting using the angles. Shown a single triangle swung back and forth (figure 8), she counted 3. At this point the instructor said: “So if there are 3 triangles, it means I can give you one, give your friend one and I will still have one for myself, right?” acting out the words. The child saw then that it was not possible, understanding that the word triangle corresponds to the physical object, not to the angle, and that perspective doesn’t change the object. This paper is not the place to discuss the ramifications of the event. Suffice to say that the instructor had a very important experience with the child.

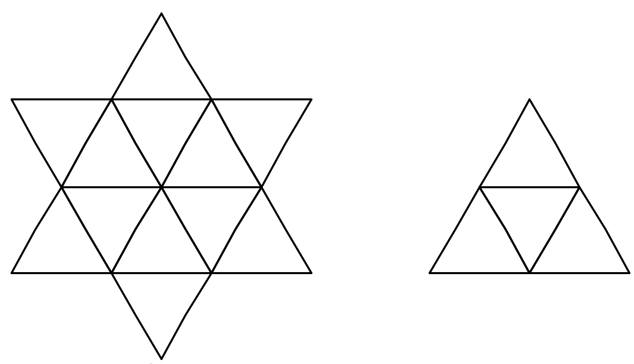

Figure 7: *The Star of David and edge-length-two triangle configurations*

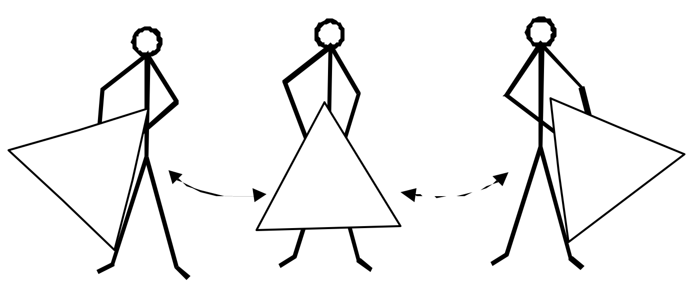

Figure 8: *Swinging the triangle back and forth*

On a more general note, this part of the lesson is still fairly straightforward. The triangles may overlap, but they can be seen as all pointing in the same two opposite direction. It is easy to count them all by enumerating the ones pointing towards the observers and the ones pointing away. In the next section, things get more lively!

- **Counting 'diamonds'.** The first ambiguity when counting what the children call 'diamonds' comes from the meaning they attach to the word. In the beginning, it appears that any convex shape in the configuration, except the triangle, is accepted as a 'diamond', even hexagons or parallelograms. This shows the need for an exact, or at least a verbalized definition that can be referred to. First, the concept needs to be reduced to a definite shape, with a specific example. For the purpose of the activity, the context specific definition is: *a set of two triangles with a common side* (in other words, a 'rhombus').

  But what is now agreed to be a diamond can have three distinct orientations in the regular triangular grid. This property, as well as the possibility of overlap, still makes them challenging to count. A convenient tool to ensure that all 'diamonds' are counted once and only once is to return to the definition: a 'diamond' is a set of two triangles with a common side, in other words, any instance of adjacency between two triangles signifies the presence of a 'diamond'. If one were therefore to count all instances of adjacency (common edges) between triangles, one would in fact have the number of 'diamonds'! Adjacency is much easier to spot. In the Star of David of figure 7, there are 15 'diamonds' (counting the small and edge-length-two 'diamonds').

- **Counting other shapes.** If the children built some three-dimensional polyhedra, there are more shapes that can be identified. For example, there are three squares in an octahedron. Again, the agreement on a definition will be important in the counting.

2.4 Grades 3-6. This lesson mainly concerns the symmetries of a single triangle and builds on the side and corner observations of the kindergarten lesson (2.2.1). The main difference at these grade levels is that the instructor can introduce more terminology. Besides 'edge' and 'angle', the terms 'vertex' and 'equilateral' can be added when the children see that the three edge-segments and the three angles are congruent. The lesson described here continues from the end of 2.2.1.

- **Finding symmetries.** Holding the single triangle with its bottom edge horizontal, the instructor asks if the children can think of a way to move the triangle so that it will look the same after it is moved. This means that where there was an edge or a vertex before, there will be an edge or vertex respectively afterwards. The instructors let each child who has an idea come forward and try it using the triangle. The solutions include two rotations and three reflections (aside from the identity).

  If none of the children can find a valid transformation, one of the instructors takes a different triangle (a clear one if possible) and places it like a mirror in front of his face so that the triangle 'reflects' the two sides of the face to each other (figure 9). Then the instructors ask the children if they can do the same thing with the colored triangle.

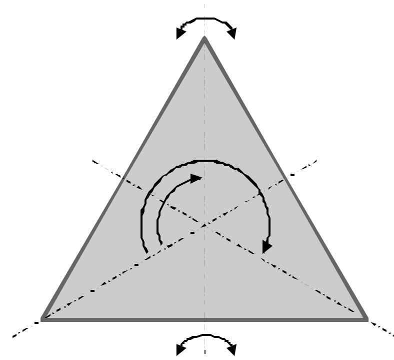

Figure 9: *Some symmetries of the triangle*

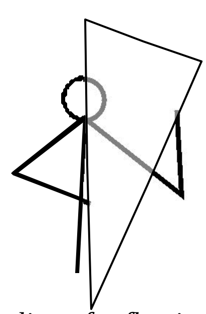

Figure 10: *The line of reflection of the face*

- **Naming the symmetry.** When one of the children finds the first line of symmetry, the instructors use masking tape to ‘draw’ the line on the triangle. The instructors point out to the children that all the points on the line remain in the same place when the triangle is reflected. The children now need to give the transformation a name. If they cannot think of one, the instructors can ask: “What do you call what you see in the mirror?” Possible answers include: shadow, me, my face, image... With the instructors’ help, the children can find the right word. The children and the instructors should then write down (on the blackboard and in their notebooks):

line of reflection

- **Finding all the lines of reflection.** The instructors now ask the children if there are other lines of reflection. The children can find two more. After the instructors have taped the two lines, everyone can modify the sentence by adding a 3 and an ‘s’:

3 lines of reflection

- **Finding another symmetry.** The instructors ask the children if they can see other movements that will preserve the appearance. If the children cannot find it, the instructors give a hint by beginning the turning motion.
- **Naming the symmetry.** In this case, the correct term, “Rotation” is probably unknown to the children. In the elementary school where this was first tried out, the children were familiar with the Spanish word ‘rotare’, but to English-speaking children it can be a new word. Rotating the triangle, the instructors ask the children to count how many times you stop at a position that conserves appearance. They will see that this happens three times. The instructors then ask the children to find the fixed points like they saw in the reflection. Using trial and error and direct demonstration, the children will see that the point in the middle of the triangle remains fixed. Under ‘3 lines of reflection’, they write:

point of rotation

- **Discussing the relationship between the symmetries.** The instructors lead the children in locating the exact point that remains unmoving through the complete rotation. When the children figure out that it is a point located on all three lines of reflection, they can complete the sentence:

The 3 lines of reflection meet at
the point of rotation.

  The children now see that not only are there two types of symmetries in an equilateral triangle, but the two are related to each other spacially. The instructors also point out that the number of lines of reflection is the same as the number of turns in the rotation symmetry.

- **Further work.** The notions of symmetry introduced in this lesson can be carried further by using assemblies of triangles such as a rhombus or a hexagon. However, taking the lesson into the third dimension by extending the symmetries of a single triangle onto a tetrahedron proved too difficult for this age group.

Conclusion

The responses of the children to these activities show that directed investigations can be used to teach mathematical concepts, at the same time as they involve teamwork, critical thinking and problem solving. This helps to develop the skills needed to use mathematics in other subjects at school and in real life applications.

In the course of the activities, the need for accuracy of language and communication arose naturally. Children have a general tendency to use the most inaccurate word they can get away with. In two dimensions, they will call rhombi as well as hexagons ‘diamonds’, and in three dimensions they will use the same word for triangular dipyramids, octahedra and even parallelepipeds. They also like to call a tetrahedron a triangle. This can lead to ambiguities in their discourse. Mathematics is particularly sensitive to this problem. Accurate use of vocabulary is not just about learning the terminology for a given topic, it is the ability to use language precicely.

Finally, in performing these activities in the classroom, the instructors have found the recapitulation of the event at the end a valuable tool. Recalling the various steps of the activity with the children gave them the opportunity to observe how well the concepts and vocabulary were grasped by the students.

[1] Morgan S. and Knoll, E., *Polyhedra, Learning by Building: Design and Use of a Math-Ed. Tool* Bridges Conference Proceedings, 2000.