Co-Counting: Demystifying Rational Number Learning

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Abstract

The aim of this article is to provide a unified framework for the ratio concept which in turn provides an alternative foundation for understanding rational numbers. In order to achieve this, a new perspective on counting is presented. Creating a conceptual hub that houses all the real numbers helps connect as well as distinguish their diverse natures and purposes. In particular, given the extent of the difficulties learners have with rational numbers, the classification proposed here may support teachers in demystifying the nature of fractions, decimal numbers, and percents by situating them in the larger scheme of sizing amounts. Indeed, the proposed co-counting structure may provide a simple yet comprehensive foundation for proportional reasoning.

Introduction

The latter half of the twentieth century and the start of the twenty-first century witnessed a steady and remarkable evolution in perspectives on mathematics education; from the New Math in the 60s to the Back-to-Basics movement in the 70s to the emergent Problem-Solving era of the 80s to the Reformed, Standard-Based approaches of the 90s, to the current Discovery-based Approach. Although the 90s are labelled the "math wars'" decade in which traditionalists opposed reformists, the frustration and the disputes over the mathematics content that ought to be taught and how it should be taught has a much longer history and continues to be highly contentious (Chernoff, 2019, Klein, 2003, Schoenfeld, 2004). Indeed, this discordance will most likely go on as mathematics educators continue their research into best practices and how to adapt them to ever changing contexts. According to the National Council of Teachers of Mathematics (NTCM), the mathematics curriculum must become considerably more focused and coherent in order to improve mathematics education (https://www.nctm.org/Standards-and-Positions/Position-Statements/Curricular-Coherence-and-Open-Educational-Resources/). Rethinking learning expectations in mathematics in terms of students' understanding of key concepts and skills has long been promoted by the National Council of Teachers of mathematics (NCTM, 2006). If improvements are to be made in mathematics education, the goals articulated at the macro level, a focused and coherent mathematics curriculum, must be reiterated at the micro level, focused and coherent mathematical content. Currently, the teaching of mathematical concepts and procedures follows a highly fragmented, disconnected, and limited scope and sequence. The traditional as well as the current curricula continue to present mathematics linearly, organized in accordance with separate strands (e.g., numbers, operations, patterns and relations, data management and probability, measurement, and geometry, being the most common) rather than in accordance with a network of interconnected core concepts and skills.

Specifically, I address the fragmented manner in which rational number teaching and learning are treated both in the research literature and in the classroom and offer an alternative perspective that aligns with the present petition for greater coherence, depth, and connectedness. According to Moseley (2005, p. 38), "Research in rational number understanding has characterized the domain of rational numbers as one of multiple and distinct perspectives that are necessary for complete understanding of the content". These perspectives reaffirm Kieren's (1976) five-subconstruct theory. This view continues to be firmly upheld in the research literature on the learning and teaching of rational numbers (Obersteiner, Dresler, Bieck, & Moeller, 2019). On this theory, students only come to understand fully the concept of fraction by perceiving it through five distinct lenses. Typically, Kieren's five subconstructs are interpreted as looking at fractions in situations when a part is compared to the whole (part-whole subconstruct), when two separate quantities are compared (ratio subconstruct), when it results from a division (quotient subconstruct), when it is a distance from zero on a straight line (measure subconstruct), and when it is a stretcher/shrinker of quantities (operator subconstruct). Pinilla Fandiño (2007) extends this list by adding seven other ways of understanding fractions: as a probability, as a score, as a rational number, as a position on a directed straight line, as a quantity of choice in a set, as a percentage, as used implicitly in everyday language. In fact, Pinilla Fandiño (2007, p. 18) claims that "it is necessary to conceptualize the fraction via all of these meanings and not just through one or two of them, a scholastic choice that would lead to failure." Lamon (2005) makes an even stronger case for upholding multiple interpretations of fractions:

> I reject the use of the word fraction as referring exclusively to one of the interpretations of the rational numbers, namely, part-whole comparisons. Because the part-whole comparison was the only meaning ever used in instruction it is understandable that fraction and part-whole interpretation has left students with an impoverished notion of rational numbers and increasingly, teachers are becoming aware of the alternate interpretations and referring to them as operator, measure, ratio, and quotient. Part-whole comparisons are on equal ground with the other interpretations and no longer merit the distinction of being synonymous with fractions. (p. 22)

Although this multiple-lens approach to defining mathematical concepts may have the intent of providing multiple experiences to learners and thus diversifying the entries into the concept of fraction, it is inadvertently leading to much confusion by blurring meaning, operations, and applications. In addition, learning fractions through this proliferation of situations in which fractions occur may result in disconnecting them from their original purpose and their place in the conceptual world of numbers.

The learning of rational numbers has been and continues to pose a significant challenge. According to Moss and Case (1999, p.122), "Although the foregoing errors are quite diverse, they all reveal a profound lack of conceptual understanding that extends across all three rational number symbolic representations and calls our existing methods to teaching these representations into serious question." Moss and Case (1999) present a classification of the explanations proposed by researchers for the difficulties students have in learning rational numbers. A first category includes those studies that claim there is too much emphasis on teaching procedures for manipulating rational numbers. As such, students do not see the fraction as an integrated magnitude, a rational number (Obersteiner et. al., 2019). A second category consists of an "Adult- vs. child-centered instruction" (Moss & Case, 1999, p. 123) in which teachers do not consider children's unprompted and intuitive attempts to make sense of rational numbers; rather, teachers promote the rote application of rules. A third category proposed is the use of representations such as pie charts that confuse rational and whole numbers[1]. Indeed, this confusion known as the *natural number bias* has been the focus of many studies (Alibali & Sidney, 2015, Christou, 2015, Ni & Zhou, 2005, Obsteiner, Hoof, Verschaffel, & Dooren, 2016, Van Hoof, Verschaffel, & Dooren, 2015). Finally, in the fourth category are the problems that arise when teaching focuses on conventional notation for expressing rational numbers in lieu of the underlying conceptual system[2]. I contend that continuing to advance a teaching strategy that assigns multiple, distinct interpretations of rational numbers adds a fifth category of important obstacles to students' learning.

Almost three decades ago, Senge (1990) cautioned against the characteristic splintered manner of approaching understanding:

> From an early age, we are taught to break apart problems, to fragment the world. This apparently makes complex tasks and subjects more manageable, but we pay a hidden, enormous price. We can no longer see the consequences of our actions; we lose our intrinsic sense of connection to a larger whole. When we try to "see the big picture", we try to reassemble the fragments in our minds, to list and organize all the pieces. (p.3)

Literature supports the view that for learning to occur, instructional design must respect cognitive architecture (Sweller, 2006, Sweller, van Merriënboer, & Paas, 2019) and that this structure happens to be a nonlinear network. "To a mind that cannot make connections, each instant is an isolated event without continuity, each thought fleeting and unrelated, each precept without relevance, each person a stranger, every event unexpected." (Greene, 2010, p. 25). Certainly, to most students, despite reforms, mathematics remains an endless sequence of disconnected and incomprehensible bits and pieces that must be learned by heart. An examination of mathematics curricula in North America demonstrate that these are organized in a manner that impedes coherence and depth[3]. As Lesh (1985, p. 439) argues, "Getting a collection of isolated concepts in a youngster's head (e.g., measurement, addition, multiplication, decimals, proportional reasoning, fractions, negative numbers) does not guarantee that these ideas will be organized and related to one another in some useful way; it does not guarantee that situations will be recognized in which the ideas are useful or that they will be retrievable when they are needed."

In this essay, I first propose a counting framework that convenes and organizes the fragments of the number concepts that are usually taught in school over several years. This framework then provides the necessary context for learning the real numbers. Second, I elaborate further with regards to one particular counting type, the one that leads to ratio concepts, which in turn specifically leads to rational numbers. I end my analysis with some concluding remarks on the teaching and learning of rational numbers.

The Counting Framework

What gives rise to the number concept is the task of sizing amounts, and this is accomplished by counting (Everett, 2017). According to studies in mathematical cognition, the ability and the need to quantify is innate (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010, Butterworth, 2005, 2018, Muldoon, Lewis, & Freeman, 2009, Nelissen, 2018). Although counting is typically and narrowly associated with the initial process of assigning a number word to each object in a collection following a one-to-one correspondence, there are other counting processes, some of which result in the development of different kinds of numbers. Specifically, I argue that counting is the foundation on which we erect all real numbers, not simply the natural ones. Learners' first experiences with numbers occur very early on when they learn to count single objects, such as the number of fingers they have on one hand. In a relatively short time, learners are exposed to new contexts for counting: they learn to size amounts in comparison to other sized amounts (ratios), such as is done when determining portions (with "half" being the most common) and they learn to size amounts relative to an origin (integers), such as is done when determining the temperature. Consequently, learning to count is learning to size amounts in these different ways (Figure 1). In this framework, "unit" is defined to be that which is counted.

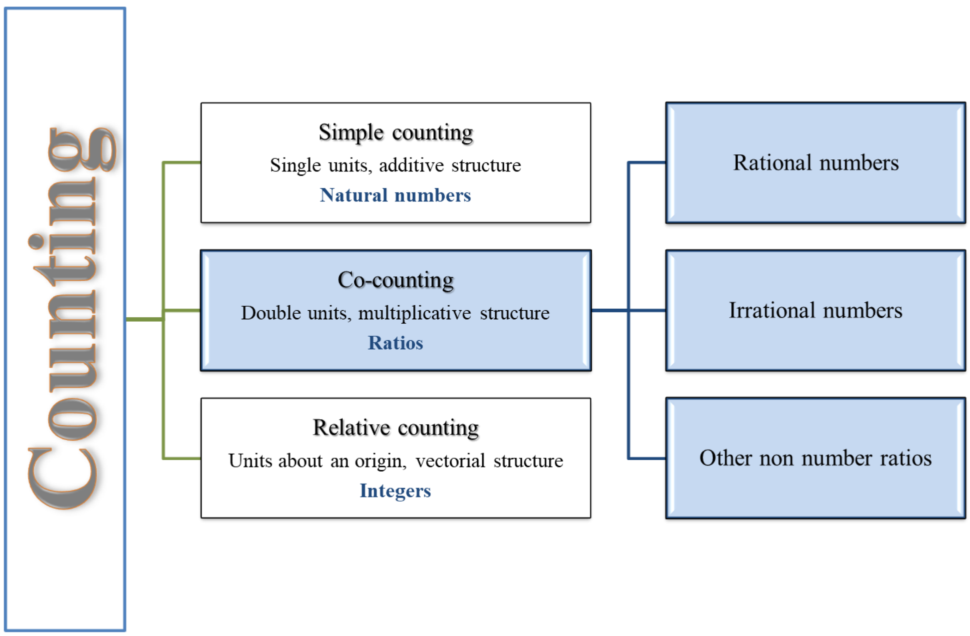

Figure 1. The counting framework

Simple counting has an additive structure because it is iteratively incremental. Note that what is counted (the unit of count) may be a whole, a portion, or a collection. For example, there could be 3 apples, 3 half-apples, or 3 bags of apples. It is this flexibility in what is counted that gives rise to the development of improper fractions (counting portions) and the elaboration of the positional numeration system (counting collections). Relative counting has a vectorial structure because the counting involves both a magnitude, obtained by simple counting, and a direction with respect to a point of origin. For example, a temperature of 3 ºC involves simple counting to 3 and doing so on a scale in the direction indicating warmth with respect to the freezing point of water expressed by 0 ºC. A temperature of -3 ºC also involves simple counting to 3 but doing so on a scale in the direction indicating coldness with respect to the freezing point of water. Co-counting has a multiplicative structure because of the relational nature of the counting. If one of the counts is increased or decreased, then the other count must be proportionally increased or decreased[4]. For example, if there is a ratio of 3 girls for every 2 boys in a class, then in a class of 25 students, there will be 15 girls to 10 boys.

According to Smith (2002, p.7), "…children's knowledge of fractions moves through two broad phases of development: (1) making meaning for fractions by linking quotients to divided quantities and (2) exploring the mathematical properties of fractions as numbers." Smith (2002) further claims that the first obstacle students face in their understanding of fractions is the need to recognize the fraction as naming the relationship between the collection of parts and the whole. In fact, co-counting is the glue the connects quotients to divided quantities. I contend that this relationship is established by *co-counting*, the counting of double units (i.e. counting together, not counting twice). Similar to counting single units in which number words are bijectively associated to objects, co-counting assigns one simple count to another simple count. For example, in the fraction two-fifths, two out of five equal parts of a whole, *two* is the simple count for the number of selected parts and *fifth* is the simple count for the total number of equal parts. Consequently, the quotient obtained, two-fifths, is linked to the quantity divided into five equal parts. Attending to both counts simultaneously is exactly what is promoted by the NCTM (2013) as the first criterion in developing a conceptual understanding of ratio relationships.

This counting framework challenges the current use of the terminology *counting numbers*, since all real numbers are counting numbers and not just the natural ones. Consequently, learners can be introduced to all real numbers jointly at a young age, allowing for a deeper understanding of counting and of number, an understanding that is foundational to learning mathematics. Once learners understand simple counting, they can directly explore its extensions to co-counting and relative counting. Existing school mathematics curricula are fragmented to the point of completely disassociating numbers from the process of counting. In addition to the fragmentation, the scope and sequence of number concepts is disarranged. For example, operations with decimal numbers is taught before operations with fractions. To understand adding decimal numbers is to understand adding fractions with the same denominator (tenths, hundredths, or thousandths). Similarly, the other operations with decimals can only be meaningful if the learner understands the operations performed with fractions. After all, decimal numbers are merely decimal fractions expressed in decimal notation. In the next section, I further develop the role of co-counting in understanding rational numbers.

Co-counting and the Rational Numbers

Although Charalambous and Pitta-Pantazi (2007) support the five-subconstruct theory for understanding fractions, they argue along with Behr, Lesh, Post, and Silver (1983) that the part-whole subconstruct is more fundamental than the remaining four subconstructs—ratio, operator, quotient, and measure. I argue that the part-whole is not only more fundamental, but essential for the other subconstructs to exist. The part-whole relationship is usually defined as a comparison between the number of selected parts and the total number of parts in the equipartitioned whole from which the parts are taken from. In other words, the part-whole relationship is a type of ratio, since a *ratio* is generally defined to be a comparison between two quantities (Lamon, 2008). Indeed, much as Confrey (1994, p. 141) criticized "that the treatment of rate in many mathematics textbooks and classrooms is ambiguous and overly narrow", the definition for ratio is often too unclear for the students to understand the relationship in question. I define *ratio* more broadly, yet more precisely, to be the result of co-counting. That is, it is the result of associating one count with another in sizing amounts. With this definition, I can organise ratios into two main categories: those that occur with respect to a single whole (unitary) and those that do not occur with respect to a single whole (non-unitary). These categories are further organized into subcategories that account for every type of ratio. Specifically, only unitary ratios form the set of real numbers. The remaining ratios are simply co-counters that do not result in any kind of number (Table 1).

Table 1. Classification of ratios

<table>
<tr><td colspan="4">RATIO
(Result of co-counting)</td></tr>
<tr><td colspan="3">Unitary
(within a whole)</td><td>Non unitary
(not within a whole)</td></tr>
<tr><td colspan="2">Commensurable</td><td>Incommensurable
Irrational numbers</td><td rowspan="3">Examples

50 km per hour

120 heartbeats per minute

Economy rate (volume to surface area)

Price per kg</td></tr>
<tr><td>Part-whole
Rational numbers</td><td>Part-Part</td><td rowspan="2">Examples

π

e

√2</td></tr>
<tr><td>Examples

½ hour
(fraction)

$ 2.90
(decimal)

30% chance of showers
(percents)</td><td>Examples

7 girls for 5 boys in a classroom (7:5)

Jug of juice is made of 3 parts juice concentrate to 5 parts water (3:5)</td></tr>
</table>

This classification broadly defines the rational numbers as part-whole ratios determined within a whole and irrational numbers as those unitary ratios determined between incommensurable magnitudes[5]. Charalambous & Pitta-Pantazi (2007, p.297) argue that “Students also need to realize what it means to say that there is a relationship between two quantities and understand the covariance-invariance property, which implies that the two quantities in the ratio relationship change together, so that the relationship between them remains invariant.” However, they never clearly define the relationship between the two quantities. I contend that defining ratios as co-counters finally sheds light on the relationship between the two quantities and explains the covariance-invariance property, which in turn explains why co-counting has a multiplicative structure rather than an additive one.

On this view, there is no need to distinguish other subconstructs of the concept of fraction as they simply become applications of the co-counter definition. In the case of the measure subconstruct, “A unit fraction is defined (i.e., 1/a) and used repeatedly to determine a distance from a preset starting point” (Charalambous & Pitta-Pantazi , 2007, p. 299). In other words, the fraction is used here as a unit of measurement, often represented by a portion of a unit line segment. This means that the fraction is not defined in terms of measurement. On the contrary, it is the part-whole ratio in which the unit is a geometrical whole that defines the unit of measure. The quotient subconstruct defines a fraction as the result of a division situation (Kieren, 1993). “Yet, unlike the part-whole subconstruct, in the quotient ‘personality’ of fractions two different measure spaces are considered (e.g., three *pizzas* are shared among four *friends*).” (Charalambous & Pitta-Pantazi, p. 299). Yet, in such situations, while the operation of division may be posed in terms of two distinct measure spaces, as in sharing, the result of the division, the quotient, is given in terms of a single measure space. That is, the sharing of three pizzas among four friends results in each individual receiving three-fourths of one pizza. In other words, the quotient is a part-whole ratio, here a portion of pizza. Finally, in the case of the remaining subconstruct, the *operator*, "rational numbers are regarded as functions applied to some number, object, or set." (Charalambous & Pitta-Pantazi (2007, p. 298). That is, the operator subconstruct defines fractions as multipliers (e.g., two-thirds of the fruits in the basket are oranges). Again, attempting to define the fraction via multiplicative applications is putting the cart before the horse. For the multiplication by a fraction to occur, the fraction must be defined. The fraction cannot be newly defined by the multiplicative situation; rather, its meaning as a part-whole co-counter is essential to perform the operation in the first place.

Defining the ratio concept as the result of co-counting provides a simple yet comprehensive framework for understanding ratios in every context. In other words, there is no need to provide a multiplicity of distinct definitions of the same concept. Rather, by understanding its essence in gathering the bits and pieces into one hub provides the basis for understanding the very nature of the concept and how it manifests itself in diverse situations.

Concluding remarks

The learning and teaching of rational numbers has been the focus of a large body of research over the past 30 years. Rational numbers are the numbers most used in daily life as well as in mathematics, but their learning has been and continues to be challenging. Davydov and Tsvetkovich (1991, p. 13) claim that the main reason for the difficulties is that "in order to understand fractions and learn the procedures involving them, one must master the mechanism of combined procedures involving not one but **two numbers**….Here the students are required to exert a somewhat greater effort of their mental powers." I add, that not only do learners need to keep in mind two numbers whenever using rational numbers, they need to master the coordination of the two numbers and understand them as unitary part-whole co-counters. Housing the rational numbers, including the integers, in the counting framework allows a simple conceptual network for learning the real numbers as well as the other ratios that form the basis for proportional reasoning.

There is no doubt that experiencing the use of fractions along with their decimal and percent homologues in diverse situations will deepen students' understanding of rational numbers as they progress through their learning of mathematics. For example, introducing rational numbers in measurement contexts rather than with slices of pizza has been shown to be more effective in developing students' understanding of fractions (Simon, Placa, Avitzur, & Kara, 2018). However, yet again, using measurement as a context for introducing rational numbers does not serve to define the numbers. Similar to children's learning of vocabulary, combining instruction in individual word meanings with instruction in deriving meaning from context has shown to be more fruitful than adopting one or the other of the approaches. In fact, "As a result of this combined program, the students could learn more than 1200 additional words in a year." (Jenkins, 1989, p.234). Learning numbers as they are presented in the counting framework reduces the cognitive load (Sweller, 2006) by making connections among the different numbers while distinguishing their various natures and purposes. The literature on learning rational numbers, and more specifically on learning fractions, has mainly focused on the distinctions rather than on the connections with natural numbers, and claiming those distinctions to be at the root of learners' difficulties. Obersteiner et al. (2019) explain that the representation of magnitude between natural numbers and fractions is an example of the differences between the two types of numbers: "The symbolic representation of natural numbers complies to the base-10 place-value structure of our number system, which allows for straightforward strategies to identify numerical magnitude." (Obersteiner et al., 2019, p.140). In contrast, for fractions, "neither number of digits nor natural number magnitudes as such determine fraction magnitudes" (Obersteiner et al., 2019, p.140). While natural number sizing may be straightforward when the learner understands the numeration system (the more digits, the larger the number is due to the underlying positional system of grouping smaller units of count into larger units of count), a parallel can be easily made regarding fractions when the learner understands the part-whole ratio (the smaller the difference between the numerator and the denominator, the larger the fraction due to the underlying equipartioning into more parts). Both require an understanding of the type of counting involved and the ways in which the numbers are expressed. Another example of a distinction provided by Obersteiner et al. (2019) is the uniqueness of the representation of natural numbers whereas fractions have several representations for a same magnitude. The unique representation of natural numbers is so only if the teaching of natural numbers is restricted to expressing numbers in base ten. While the symbolic representation of "6" is unique in numeration systems in bases 7 and up, it is not so in lower bases: 110_{two}, 20_{three}, 12_{four}, etc. In fact, if numeration was taught as a system in general, it would facilitate understanding the symbolic expression of both natural numbers and fractions. Learners could then see that a whole magnitude expressed in various ways, depending on the size of the groupings, parallels a partial magnitude being expressed in various ways, depending on the size of the parts.

According to Reigeluth (1999) and van Merriënboer (1997), focusing first on the building blocks on which more complex ideas and tasks are erected provides learners with a snapshot of the whole idea or task before further elaboration. Indeed, such a snapshot acts as an anchor for further learning. It is the initial text to which all other hypertexts are linked. For numbers, knowing the counting framework provides learners with the backdrop for understanding the ratio framework, which in turn provides them with the building blocks needed to understand rational numbers. Borrowing Byers' (2007) binocular vision metaphor for ambiguity, the world appears flat when looking out of one eye; however, looking out with both eyes allows depth perception, which provides an entirely new perspective on the world. In the same way, looking at numbers as the result of counting single objects is like looking at the world through one eye, while looking at all of the numbers through both eyes as resulting from the various counting processes opens the door to a richer and deeper understanding of the number concept.

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[1] For example, depicting fractions as slices of pizza is confusing since usual discourse during the sharing of pizza involves natural numbers, not fractions. We say we ate 1, 2, or even 3 slices; we do not way we ate, 1/8, ¼, or 3/8 of the pizza.
[2] The fraction is defined in terms of its symbolic expression: A fraction is a rational number expressed in the form *a/b*, where *a* and *b* are integers numbers and $b \neq 0$.
[3] For example, the Common Core Standards in the United States (http://www.corestandards.org/Math), and the curricula in Eastern, central, and Western Canada are organized according to a sequence of topics in distinct strands.
[4] Two rational numbers, *a/b* and *c/d* are proportional, if and only if $ad = cd$.
[5] Commensurability means that for any two magnitudes, there is standard unit that fits some whole number of times into each of them. In other words, the two magnitudes share a common unit of measurement. Not all magnitudes are commensurable. For example, the length of the diagonal of a square is incommensurable to the length of its sides (See Euclid's Book X, https://mathcs.clarku.edu/~djoyce/java/elements/bookX/bookX.html#defsI)