Polyhedra, Learning by Building: Design and Use of a Math-Ed. Tool

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Abstract

This is a preliminary report on design features of large, light-weight, modular equilateral triangles and classroom activities developed for using them. They facilitate the fast teaching of three dimensional geometry together with basic math skills, and create a lasting motivational impact on low achievers and their subsequent performance in math and science.

In directed discovery activities, lasting from 20 to 90 minutes, large models of basic polyhedra are made, enabling their properties to be explored. Faces, edges and vertices can all be counted and tabulated, providing opportunities to see number patterns and inter-relationships, to plot graphs, to extract algebraic relationships and to look for proofs of those relationships. These building activities can be kept central, under the teacher's control for large classes with limited time, or building can be split out into groups of children where co-operative problem solving skills are also developed.

In interviews, children have stressed the effectiveness of learning by building the shapes themselves. In classroom activities, it is clear to see that these triangles make children excited. Learning by building gives a concrete, active, authentic and personal experience of mathematics to children and teachers enabling them to feel the full excitement of the subject.

Introduction

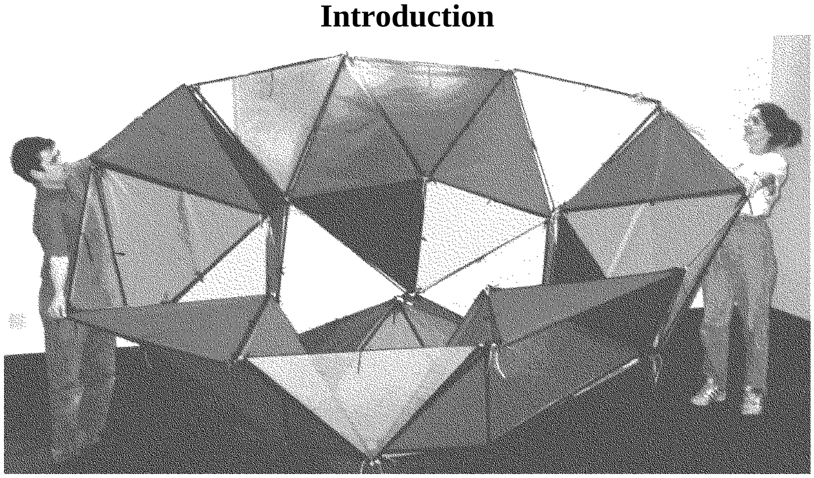

Figure 1: Eva Knoll and Simon Morgan holding half an endo-pentakis-icosi-dodecahedron., Montreal, March 1999

This is a preliminary report on an educational project which begin as an exercise in the combining of mathematics and art. Eva Knoll developed a way of folding an origami grid of equilateral triangles from a disc of paper, and then constructing polyhedra from this grid [2]. With the help of Simon Morgan and Roger Tobie the same technique was used to make a 1 m diameter 80-sided shape. The experience of this construction process convinced us to go ahead and build larger triangles (see figure 1) so that a whole group could participate in a barn-raising event [1] and [3]. To take the project further, and with the support of Rice University, these ideas and materials were taken into a Houston middle school to be tried out in mathematics lessons.

The first thing that struck us was how the size, colors and construction of the triangles excited the children and made them to want to build three dimensional shapes. In the classroom, this translated into the triangles' effectiveness as teaching and learning tools for geometry and a way of reaching and motivating low performers. The simple modular construction of equilateral triangles makes them an extremely versatile tool for teaching different concepts in a short space of time and creates endless possibilities for building shapes of a scale which really appeals to children. The educational applications have grown steadily since then, including workshops for teachers as well as deeper mathematical investigations in geometry and topology for the classroom.

Description and design features of the triangles

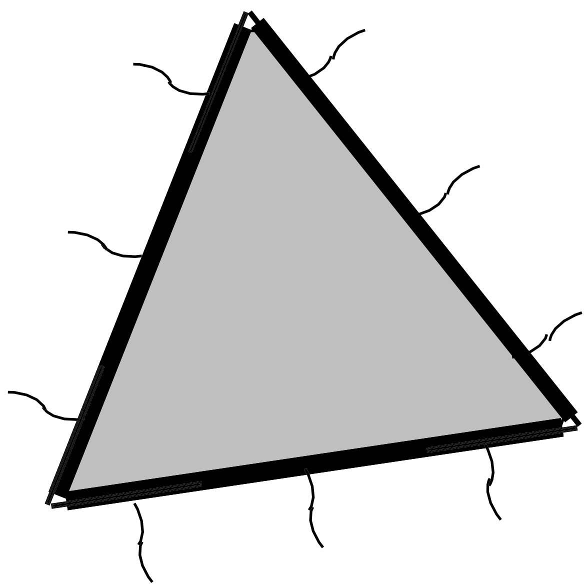

Figure 2 A triangle

The patented equilateral triangles are made of kite materials: carbon fiber rods, nylon and clear Mylar sheeting. Laces are attached to each edge so the triangles can be fastened to each other (see Figure 2). Each of three rods is enclosed in a black seem running along an edge of the triangle. The rods are hollow allowing a cord to be passed through each rod and tied fast to give the triangle its rigidity. This design was motivated entirely by the need for modular triangles which could be used in group participation sculpture making events (see phase 2 below, [1] and [3]). Here are the key design features which were identified as important for educational effectiveness in the classroom:

a) Size is the big attraction. We made the triangles as large they could be built, 1 m edge length. Children were immediately excited by that size because it enabled them to get inside the shapes. This experience of being able to see polyhedra you have built yourself from the inside and the outside stimulates the mind in a way that cannot be achieved on a hand-held scale or by looking at pictures or computer simulations. The size also adds versatility to classroom activities so that either the whole class can watch shapes being made, seeing every detail, at the front of the class, or children can work separately in groups where it is easy for the teacher not only to see what they are doing, but also to show the whole class the different results.

b) No right angles! The departure from right angles takes children away from the familiar rectangular world of cuboid rooms, buildings and objects. Every shape made looks unusual in an unpredictable way and therefore provokes curiosity.

c) Light weight materials give versatility in the classroom. The light weight hi-tech materials enable the polyhedra to be built safely, easily and quickly using lace ties. They also ensure that the polyhedra are attractive and rigid with a clear geometric shape. Fully assembled polyhedra can easily be picked up and moved around by a teacher or a child. This helps children develop a tactile feel for their shapes as well as allowing the shapes to be held up and spun around to show how they look from different angles, and to demonstrate their symmetries.

d) Modularity gives simplicity, speed, versatility, full utilization and demands problem solving. Having just one equilateral triangle modular component coming in a range of colors makes for straightforward construction of an unlimited number of fascinating shapes, with no time consuming setting up of materials. Any child can continue building any shape, using any of the triangles, until they are all completely used up. The flexibility of the how triangles can be put together forces children to make choices and to decide how to build a shape, either in free form experimentation or in trying to build a polyhedron with specified properties.

e) Lace ties enable speed and discovery learning. These are great for a fast moving class. They keep the instructions simple as everyone knows how to tie shoe laces. Three ties are needed for one edge, so it does not take too long for a group of children to put a large shape together. They just want to keep building more and bigger shapes. Speed of untying is also critical for this classroom tool. It enables many shapes to be built and modified, one after the other, re-using the same triangles and enables fast dismantling and storage at the end of the class which saves teacher time.

f) Rainbow colors attract interest and dispel fear. Using deep saturated rainbow colors adds to the triangles' immediate appeal and emphasizes the contrast between adjacent faces on polyhedra. Any fear of mathematics tends to be forgotten as children decide which is their favorite color of triangle to start building with.

g) Clear triangles show the inside and the outside at the same time. Clear triangles are like windows into another world, the inside of a polyhedron. If a polyhedron is made entirely of clear triangles it is possible to see all the faces, edges and vertices at once, which helps emphasize its three dimensionality.

h) Black ribbon edging gives geometric definition to shapes. There is a 1/2 inch black ribbon bordering each triangle. This marks out clearly the edges and vertices of polyhedra which in turn demarcate the distinct faces and their orientations.

i) Rigid rods in each edge give tactile and conceptual understanding. Rods inside each triangle edge under the black ribbon have a tactile quality, they are straight, rigid and what you feel and hold when you pick up a triangle. This direct tactile experience of edges adds to the child's understanding of polyhedra. The rods are useful when explaining what happens to edges when triangles are assembled into polyhedra. A triangle by itself has one rod per edge, but in a polyhedron two triangles come together along an edge giving two rods per edge. If we pick up a polyhedron by an edge we can actually feel the two rods in that edge, giving a concrete experience that precedes or reinforces the abstract concept (lesson 4).

Classroom activities

We explain the lesson plans developed so far in chronological order. Additionally we have often allowed children to build whatever they want with the triangles. This free form work has involved a lot of variety, learning and problem solving which encourages children to think for themselves. Phases 1 and 2 took place together in one semester culminating a big event [3]. Phase 3 followed as an attempt to develop more mathematical content for lessons with the triangles. The work with low performers in phase 4 was done in response to seeing successes with individual 'at risk' children who used the triangles and to address the special needs of this group. Free form work and lesson 3 make good starting points for students at all levels who are going to use the triangles.

Phase 1: Platonic solids. These early activities were designed to help the children experience the geometry of solids through building, seeing and physically getting inside shapes. They took place in a high school geometry class and with regular 8th grade students in Lanier Middle School, Houston, TX.

Lesson 1: Filling tetrahedra and the stella octangula. This lesson involved constructing a large tetrahedron with 2 m edge length and four triangles making up each face. This gave a chance to review how volume and surface area scale with length. The children were asked how many 1 m edge length tetrahedra could fill the inside. After four tetrahedra were placed inside, a space was left that was filled by another mystery shape that the children constructed with triangles.

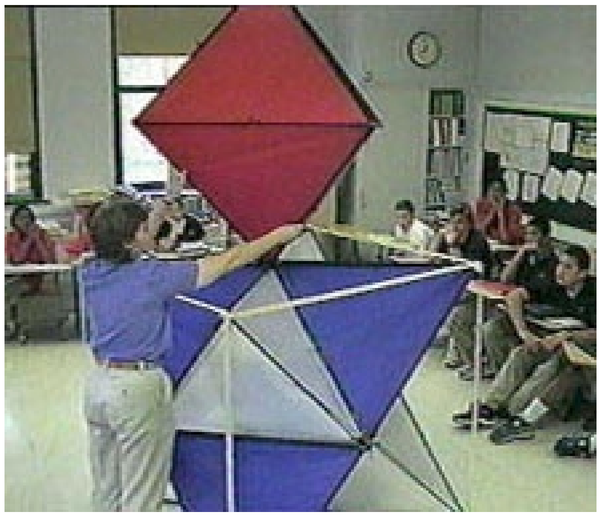

Figure 3 Tape reveals the cube

Figure 4 Building icosahedra

Finally, to continue the space filling theme the middle triangle of each face was stellated to produce a stella octangula (see figure 3). Another stella octangula was made by stellating an octahedron, and then the two stella octanguli were placed so that they enclosed an octahedron, demonstrating how to fill 3 dimensional space. The key excitement points in this activity were trial and error with filling the large tetrahedron and looking inside the stella octangula during construction.

Lesson 2: Building icosahedra. The children built two icosahedra, one hand held by folding a paper disc [2], and a scaled up version using the triangles. The large version icosahedron had to be colored with 5 colors so that no two triangles at a vertex had the same color. The children achieved this by planning their coloring on the paper nets they had used for the small icosahedron, laying out big triangles in a net following that plan, and then closing up the giant net just as they had done with the paper models (see figure 4). The key excitement points were having a paper icosahedron that unfolded into snowflake shape that they could take home and getting inside their full size icosahedra.
Phase 2: Geraldine. Barn-raising an endo-pentakis-icosi-dodecahedron (see website video [3] and [1]).

Phase 3: Theorems and proofs by directed discovery. For group work, a set of step by step instructions was prepared, each of which required some problem solving e.g. build a closed shape with four triangles. For each step there were some tabulated records to be made, such as writing down the number of edges or vertices, or drawing the shape. The adult instructors went round and checked that each building step had been done correctly and that each child had written down the observations. For low performers we sometimes adapted this plan in phase 4. At various stages the class can be brought together to examine the different shapes made and to look for trends, to plot graphs, to extract algebraic relationships and to discuss proofs of formulas or theorems derived from experiment. As much as possible the children are asked to suggest things and come up with their own reasons rather than just being told the answers.

Lesson 3: Making n-gon based pyramids. This lesson was designed as a preliminary lesson to introduce children to the difference between the flat plane and a pointed vertex of a polyhedron. They count six triangles that would come together flat on the floor, and then see different pyramids produced as they reduce the numbers of triangles coming together at a vertex. Then they are asked to make closed up shapes with a fixed number of triangles at each vertex, i.e. three to make a tetrahedron, four for an octahedron and five for an icosahedron. For gifted children this lesson can combine with lessons 4 or 5. See phase 4 for enriching this lesson for low performers.

Lesson 4: Combinatorial topology (5th grade and above). The simplest formula involving a polyhedron with all triangular faces is 3 x F = 2 x E where F is number of faces and E is number of edges. This can be discovered by building shapes with a fixed number of triangles and counting the edges. Tabulating and graphing the number of edges for the different number of faces can enable the children to find a formula relating the quantities. Children also discover by experiment that you cannot make a polyhedron with an odd number of triangles. Finally we ask why is the formula true? Then we observe that for each triangle there are three rods in the edges of one triangle, and in each polyhedral edge there are two rods. So we see we are counting the total number of rods in two ways, as three times the number of triangles or two times the number of edges in the polyhedron (Rods = 3 x Faces = 2 x Edges). Then we see that if we have an odd number of faces, we would get a half edge, thus proving (by contradiction) that it is impossible to make a polyhedron with an odd number of triangular faces.

Lesson 5: The angle deficit theorem (7th grade and above). The Euler formula for a polyhedron with no holes (faces - edges + vertices = 2) has a geometric counter part, the angle deficit theorem. Angle deficit at a vertex can be measured by adding all the angles at the vertex and subtracting that from 360 degrees. For example, a vertex of a cube has three 90 degree angles, giving a total of 270 degrees. Subtract this from 360 and we get a deficit of 90 degrees. The theorem states that the sum of all angle deficits at each vertex on a polyhedron adds up to 720 degrees, or two whole turns. A proof of this (not part of the lesson) can be derived from the euler characteristic of a sphere and the fact that the sum of the interior angles of a triangle is 180 degrees. Also see lesson 6 for another proof.

The class activities involve building tetrahedra, octahedra and icosahedra, counting and tabulating the angle deficit at each vertex (recorded in fractions of 360 degrees) and the number of vertices. The children then are asked to find a number pattern (see table 1).

Angle deficit (Pointiness)	no.of vertices	Polyhedron
1/2	4	Tetrahedron
1/3	6	Octahedron
1/6	12	Icosahedron

Table 1

They notice that the number of vertices is always twice the denominator in column 1. They are shown that this means the product of angle deficit and number of vertices is always two. We also discussed why the number of vertices increases as the angle deficit decreases. Basically angle deficit measures how pointy or sharp a vertex is, which in turn determines the angles between faces meeting at that vertex. To close up the shape it is necessary for the faces to go all the way around the shape. Pointy vertices (e.g. on tetrahedra) turn the faces around more than less pointy vertices (e.g. on icosahedra). So if you have vertices that are less pointy, or flatter looking, you need more of them to get the faces all the way around the shape. These explanations become more meaningful when the teacher is holding up and comparing giant sized tetrahedra and icosahedra. Physically pointing out the shapes of vertices and the angles between faces around them as the explanation progresses makes it completely clear. Then the angle deficit formula is stated, explained and verified where possible by the children for other shapes such as pyramids, prisms, the cube and the dodecahedron.

Lesson 6: Non-euclidean geometry (9th grade). This lesson starts with the children investigating how a geodesic or shortest path goes over an edge of a tetrahedron. Then three points (a,b,c on the left of figure 5) are chosen around a tetrahedral vertex and connected by masking tape geodesics so that a vertex of the tetrahedron is enclosed within this ‘geodesic triangle’. The children measure the sum of the internal angles of the triangle and find that they add up to 360 degrees instead of the usual 180 degrees for triangles in the plane. The enclosed tetrahedral vertex is then opened up by untying along one edge of the tetrahedron. See figure 6 where three triangles around a tetrahedral vertex are opened up and laid out flat to examine the angles. The geodesic triangle going around through points ‘a’, ‘b’ and ‘c’ on the left (and also ‘e’ and ‘f’ when flattened out on the right) is marked by the children with masking tape.

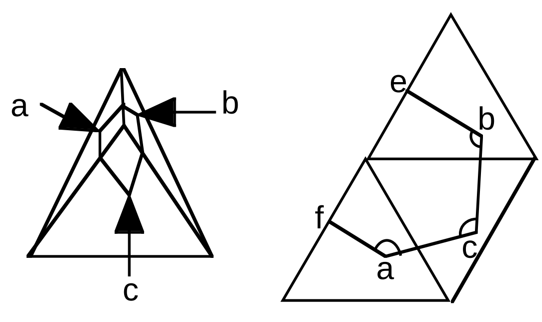

Figure 5 Tetrahedron with a geodesic triangle ‘abc’ (left) becomes pentagon ‘facbe’ after the enclosed tetrahedral vertex is opened up by untying (right).

One group of children then tried to prove mathematically that their protractor measurements were accurate. One child proved this himself and his proof was shown to the rest of the class on the triangles that had been unfolded, as in the right hand side of figure 6. Pentagon ‘abcdef’ has a sum of interior angles of 540 degrees (as it can be decomposed into three triangles) and as ‘eb’ is parallel to ‘af’, the angles at ‘e’ and ‘f’ must add up to be 180 degrees, leaving 360 degrees for the sum of the angles at ‘a’, ‘b’ and ‘c’.

The general formula for the sum of the interior angles of a triangle on a polyhedron is 180 degrees plus the angle deficits of the enclosed vertices. This can also be used to prove the angle deficit theorem by considering two complementary triangles bounded by a given set of three geodesics. This lesson can be varied by repeating the exercise using octahedral or icosahedral vertices and by allowing a geodesic triangle to enclose more than one vertex.

Low performing children

Phase 4: Remedial work. We have carried out work with remedial 6th grade through 12th grade groups taking from all the above lesson plans and allowing free form work. One 6th grade remedial teacher used the triangles exclusively with free form work to encourage independent thinking and group work. The children discovered a wide variety of three dimensional shapes for themselves.

Additionally we used the triangles for teaching and reviewing two dimensional geometry and use of protractors. When making pyramids (lesson 3), we place masking tape around their bases on the floor, then remove the pyramids and see what polygon emerges. Then we can measure and calculate the interior angles. Masking tape is used to triangulate the hexagons and pentagons that emerge, and the properties of the triangles can be discussed, measured and proven. Doing geometry ‘life-size’ on the triangles and on the floor with masking tape and chalk is an effective strategy to elicit student participation in the geometry proofs and discussions. Children are more likely to make suggestions and ask for something to be explained again until they understand it when working with large manipulatives. It is valuable to be able work a proof or derivation in different ways, so whenever students make suggestions the teacher can help them to develop the line of reasoning. If a class then sees many different proofs of the same thing based on student suggestions, then all the better.

With low performing children, in contrast to high performers, it is best to ask each group to build a different shape and keep a table of results on the blackboard, rather than having each child keep their own record as each group builds every shape. Low performers can complete the whole of lessons 1, 3 and 4, but more time is needed when examining number patterns in data, plotting graphs and seeing how to derive equations. When lesson 3 is done with protractor work, low performers can then go on to do part of lesson 5.

Two key pedagogical aspects of working with low performers in mathematics are diagnosis of deficiencies in mathematical basics and then teaching with ‘hands-on’ manipulatives. These activities certainly provide the hands on manipulatives that these students need, and in addition provide teachers a chance to see what basic math skills their low performers have. We have found with the number work that teachers can be surprised by their students, not just by seeing gaps in ability, but also by finding that some students had abilities that had never previously emerged. Just as important as these pedagogical aspects are long term motivational improvements in the children that have been observed.

Discussion

Using this design of triangle gets children excited about mathematics and teachers have found that these activities can completely change the attitude of children who had been low performing and described by the teachers as ‘disconnected from school’. The children have stressed how their understanding was helped by building shapes themselves. In the video [3] from Phases 1 and 2, teacher Jackie Sack says:

> “when kids are exposed to something of this size that they can feel and touch and move around then they truly begin to understand it, understanding with a sense of excitement and

curiosity and this is impossible to achieve on a tiny hand held scale. Once you have this excitement and curiosity you do and discover all kinds of things."

This comment seems also to apply when we later broadened the scope of the mathematics covered in the lessons. The concrete real world nature of these shapes makes mathematics real and relevant to children in a way working on paper may not. As the children are working with triangles, seeing how to put them together, having ideas, trying things out, being mathematically creative and innovative (particularly in free form work), and getting immediate results all the time, they see a very different view of mathematics. This is not a view constructed step by step by a teacher or a book, it is not seeing someone else's explanation of mathematics, it is seeing by doing it oneself. Learning by building gives a concrete, active, authentic and personal experience of mathematics to children and teachers enabling them to feel the full excitement of the subject.

These building and discovery learning activities that go beyond the syllabus have proven to give benefits to the children's learning and use of basic syllabus material that more than justify the time spent on them. These activities are full of opportunities to review and apply arithmetic and algebra skills and two dimensional geometry. The scientific method is also integrated into this process by use of experiment, making, tabulating and graphing observations, seeing patterns in data, trying to define and explain them mathematically, and then going on to discuss mathematical proofs of observed trends.

Results attained so far justify large scale educational research into the advantages of using these materials and activities, both in terms of geometry and basic math skills and also in terms of motivating underachievers in mathematics in a lasting way. For school classrooms, we recommend having a central project for each district which has a set of triangles and one or two people to go out to schools to use them with children. Schools that use the triangles a lot can buy their own set. For museum displays, we recommend having shapes you can get inside and organized shape building activities for visitors school groups. The authors are available to supply triangles and lesson plans.

References

[1] Eva Knoll and Simon Morgan. *Barn Raising an Endo-pentakis-icosidodecahedron.* In Bridges: Mathematical Connections in Art, Music, and Sciences. Conference Proceedings 1999. Reza Sarhangi, Editor.

[2] Eva Knoll. *From Circle to Icosahedron.*In Bridges: Mathematical Connections in Art, Music, and Sciences. Conference Proceedings 2000. Reza Sarhangi, Editor.

[3] Website video: http://math.rice.edu/~rusmp/