From a Subdivided Tetrahedron to the Dodecahedron: Exploring Regular Colorings

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Abstract

The following paper recounts the stages of a stroll through symmetry relationships between the regular tetrahedron whose faces were subdivided into symmetrical kites and the regular dodecahedron. I will use transformations such as stretching edges and faces and splitting vertices. The *simplest non-adjacent regular coloring*[1], which illustrates inherent symmetry properties of regular solids, will help to keep track of the transformations and reveal underlying relationships between the polyhedra. In the conclusion, we will make observations about the handedness of the various stages, and discuss the possibility of applying the process to other regular polyhedra.

Introduction

The symmetry relationships between the Platonic solids are well known, as are their simplest non-adjacent regular colorings. These properties can in fact be used as basis for many activities, including their construction and observations about their properties. The present example began with a particular coloring of the tetrahedron illustrating some of its more sutle properties of regularity, and ended up somewhere completely unexpected. The entire exploration is presented here in a narrative because the order of the events adds a significant clue as to its richness. Derived from activities in polyhedral Origami and mathematics education, the exploration gives a good example of discovery through exploration.

The experiment

2.1 The Premise. The entire experience began with the following question:

> *What is the simplest non-adjacent regular coloring of a tetrahedron whose faces have been subdivided into sets of three kites (see figure 1)?*
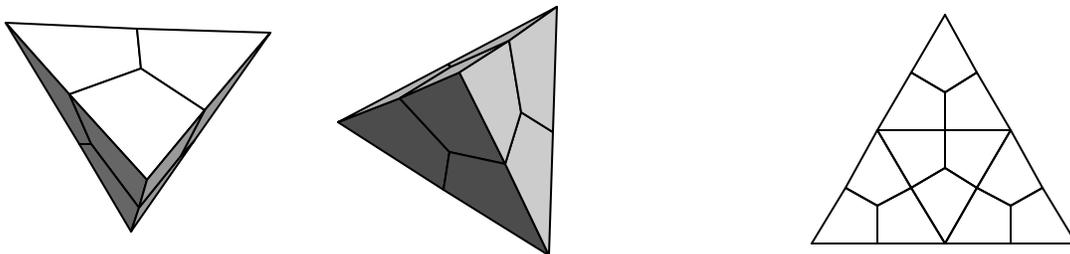

Figure 1: *Two views of the tiled tetrahedron and one of its possible nets*

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[1] In this paper, the simplest non-adjacent regular coloring is defined as the coloring using the least number of distinct colors where no two same-colored regions are adjacent and each set of regions of the same color is isometric to the others on the surface of the polyhedron.

2.2 The Simplest Non-adjacent Regular Coloring. This tiling of the tetrahedron conserves all the symmetries of the tetrahedron that supports it. Additionally, it contains twelve 'faces', twenty-four 'edges', six occurences of 4-vertices (at the mid-point of each edge of the supporting tetrahedron) and eight occurences of 3-vertices. The latter are subdivided into two groups: four at the vertices of the supporting tetrahedron and four at the centers of its faces. The simplest coloring, therefore, must contain at least three colors (because of the 3-vertices), but needs no more than four, even if we require that all the 'faces' remain equivalent (see figure 2). This is a very important condition, leading us to the next step.
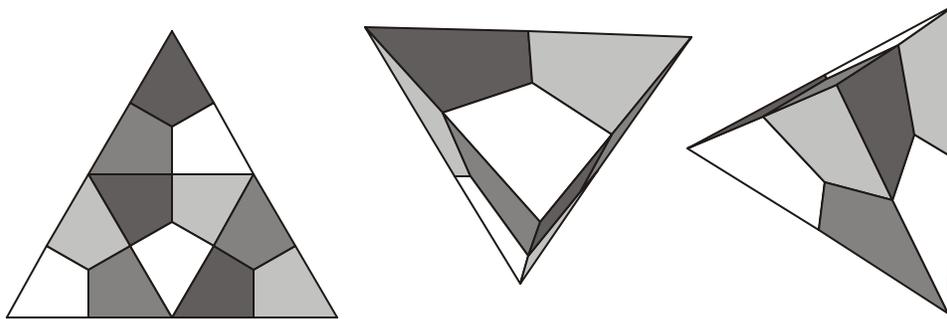

Figure 2: The simplest non-adjacent regular coloring of the tiled tetrahedron

2.3 Coloring the dodecahedron. If there are four colors and twelve faces, we can say that:

a) Each color is used three times.

b) The number of colors and their frequency are the same as in the regularly colored dodecahedron.

This is interesting, and worthy of further exploration: Consider the simplest non-adjacent coloring of our shape (figure 2) and the simplest non-adjacent coloring of the regular dodecahedron, which also contains four colors each used three times. If there is a symmetry link between the two colored polyhedra, it should be possible to create an animated sequence moving one to the other without changing the colors.

Let us further compare the tiled tetrahedron and the dodecahedron, counting each kite as a 'face':

Definition	'Faces'	'Edges'	Vertices	3-vertices	4-vertices
Tiled tetrahedron	12	24	14	8	6
Dodecahedron	12	30	20	20	0

2.4 Adjusting the Properties. Looking at the number of 'faces', things appear fine, but we seem to have a discrepancy in the number of 'edges', and once we reach the vertices, all seems definitely lost. Not only do the numbers not match, we don't even seem to have the right *type* of vertex! If we could somehow transform the six 4-vertices into twelve 3-vertices, and if the number of edges were then also adjusted, we would have something to work with.

One way to transform 4-vertices into 3-vertices is to stretch the vertex by introducing an edge between two pairs of incoming edges, as in figure 3.
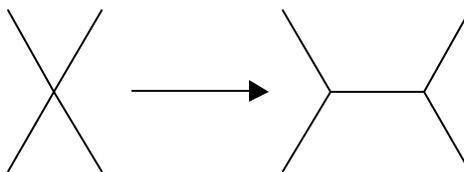

Figure 3: Transforming a 4-vertex into two 3-vertices

In figure 4, we can see the triangular net of figure 1 modified accordingly. The new edges are emphasized in the diagram on the right. Since the figure is meant to closed, the new edges on the perimeter of the net are in part the same, and therefore must be counted only once.
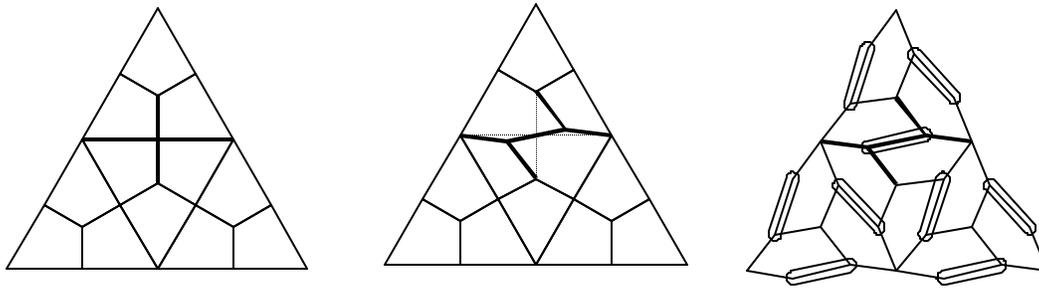

Figure 4: Transforming the net

We can now add a new line to the table as follows:

Definition	'Faces'	'Edges'	Vertices	3-vertices	4-vertices
Tiled tetrahedron	12	24	14	8	6
Stretched tiled tetrahedron	12	30	20	20	0
Dodecahedron	12	30	20	20	0

Topologically speaking, we now have the correct combination of 'faces', edges and vertices. There are of course many possible positions for these new vertices. Let us have a look at some significant ones:

a) The new points can be placed somewhere along the edge of the supporting tetrahedron. These solutions are interesting because they conserve the existing coplanarities among the 'faces', preserving the supporting tetrahedron.

b) The new points can be placed in such a way that all the edges in the net are of the same length.

As it turns out, there are two solutions that conform to both of these conditions simultaneously. The two solutions are mirror images of each other, but if they are combined with the coloring, which already possesses handedness, they become distinct. Figure 5 shows both the nets and the closed shape (including the coloring) for the two solutions. The net in the middle corresponds to the original subdivision, and the ones on the right and left sides show the new nets with the 4-vertices transformed into 3-vertices, all 'edges' being of equal length. Note that the 'faces' of each color retain their relative positions.
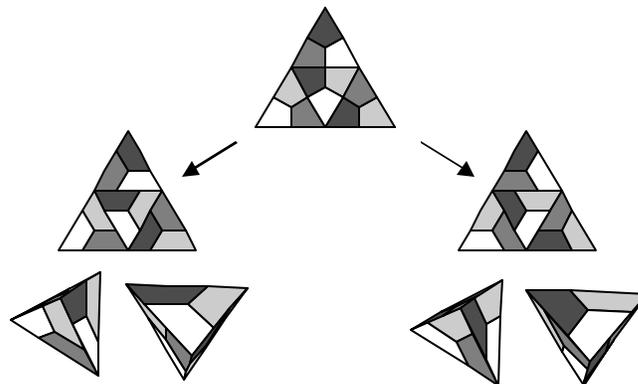

Figure 5: Transforming the tiled tetrahedron

2.5 The Transformation. Once the tetrahedron is tiled with the new faces, it is time to look at the transformation leading to the dodecahedron. Since all the edges are already of equal length, only the angles remain to be adjusted. Figure 6 shows the entire sequence from the kite-tiled tetrahedron to the stretched tiled tetrahedron, to the regular dodecahedron. Throughout the whole sequence, the four vertices of the supporting tetrahedron do not move, and in the second part, the vertices on the faces and edges pull out to become equivalent vertices on the dodecahedron.
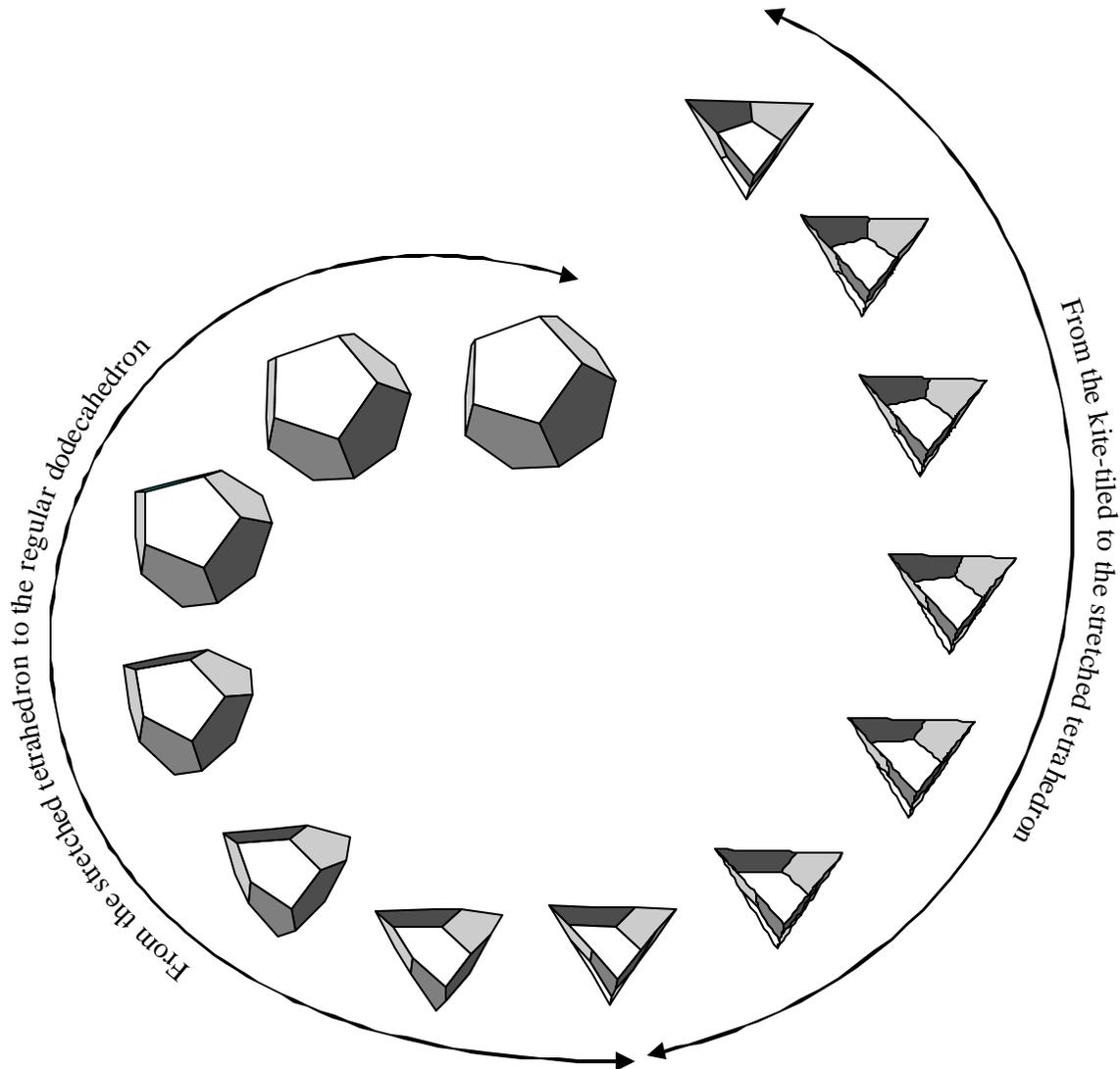

Figure 6: transformation from the kite-tiled tetrahedron
to the stretched tiled tetrahedron, to the dodecahedron

Conclusion

3.1 Handedness. The series of procedures described above began as a stroll without prescribed destination. It lead us through interesting developments, each of which showed something about the Platonic solids and some of their derivative polyhedra. In several of these steps, the transformation introduced new factors, new symmetries, and new variations. In particular, at several stages the

transformations introduced handedness, meaning that there were really two separate solutions that were each other's mirror image. In most cases, this double solution can be counted as only one, but if handedness is introduced at more than one stage of the transformation, this may not be the case. In the present exercise, the handedness was introduced by the coloring, first, and then by the vertex stretching of figures 3 and 4. The handedness introduced in the first case gives two solutions that are in fact each other's mirror images. In the second assymmetric transformation, the stretching of the 4-vertex into two 3-vertices, the handedness that is introduced can be applied in the same direction or in opposition to the first one. Because of this, there are then four solutions, which we will call the left-handed/left-handed, left-handed/right-handed, right-handed/left-handed and right-handed/right-handed solutions. If we discounts the mirror image solutions in the present situation, there are only two solutions, one with same handedness in both transformations, and one with opposite handedness.

3.2 The Other Platonic Solids. Although the transformation from the kite-tiled tetrahedron to the dodecahedron was developed as a byproduct of the coloring problem, it demonstrates beautifully the spacial relationship between two Platonic solids that are not each other's duals. This brings up interesting possibilities for further investigation: Can there be similar coloring-conserving transformation between some other pairs of Platonic solids? The structural relationships between the five polyhedra, as well as their intrinsic properties, are well known, so the problem is not impossible to solve, but the result will not necessarily be as aesthetically pleasing as the tetrahedron-dodecahedron relationship.