

Circular Origami: a Survey of Recent Results

Introduction

For many years now, I have been studying systems of constraints in different design media. These studies in turn fuel my own creativity and inspire me to produce artwork and theoretical papers in both art and mathematics. My inspiration comes from many different media, including Islamic tiling, Celtic interlace, Japanese paper folding traditions, modern art, particularly the Constructivist and Concrete movements, even natural phenomena and effects of perception.

In this particular instance, I will report on my experience so far using circular paper for Origami. This paper was deliberately separated into three sections, starting with a short chronology of the discoveries I made in circular Origami. The middle part of the paper describes the mathematics that I derived from these explorations. Finally, the last part reports on the results achieved in mathematics education through the use of my discoveries and products derived from them.

Circular Origami

The story of my explorations in circular Origami begins with the coincidence of two encounters I made in the summer of 1996. It was that summer that I met Chris Palmer, who through his work reintroduced me to Origami and demonstrated its exploratory aspect. It was shortly thereafter that I came across paper of appropriate quality for folding that was cut into circles. What a novel idea, and what an interesting set of parameters!

At first of course, I used my circular paper like a square, folding it into 90 and 45 degree angles. This proved interesting only to a point. For one thing, the bird base and water-bomb base amount to the same thing in circular paper (see figure 1).

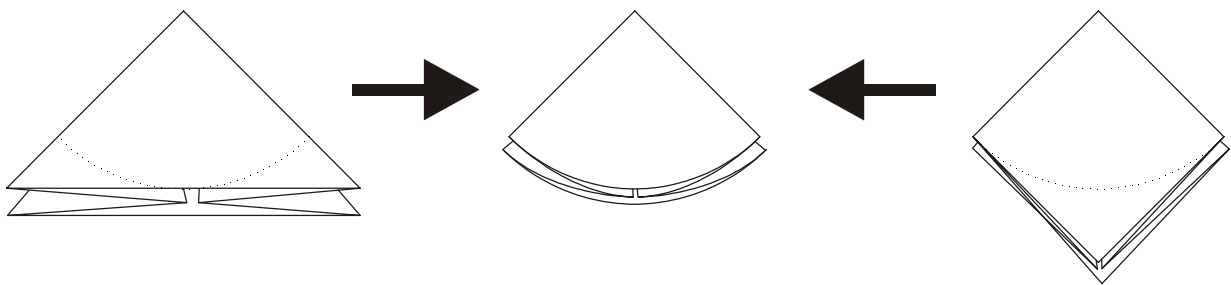


Figure 1: Same result from bird and water- bomb base

But the circle has some unique properties. Besides having infinitely many axes of symmetry, it lends itself well to certain applications of trigonometry. This might seem obvious in light of the fact that trigonometry is based on the geometry of the circle, but the results in this case are stimulating. In figure 2, a fold is made that introduces an unexpected angle of 30 degrees without having to resort to measurement, approximation or drawing. This is of course due to the fact that $\sin 30^\circ = 1/2!$

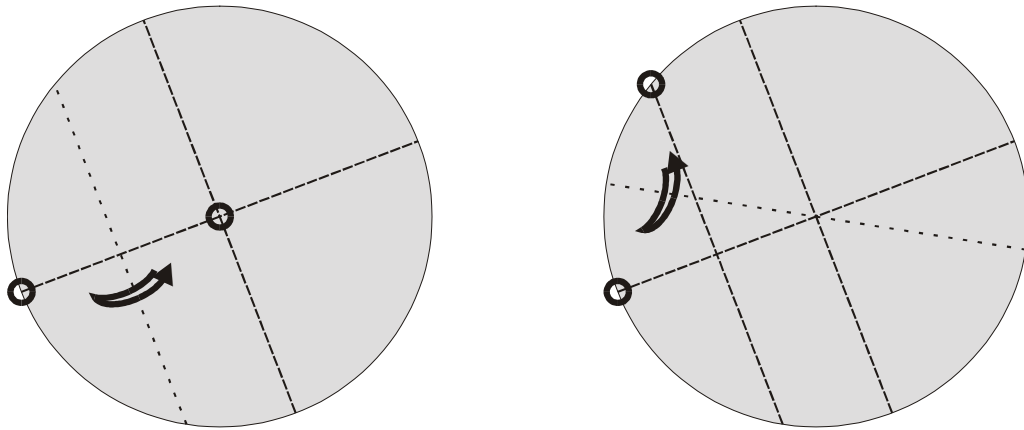


Figure 2: $\sin 30^\circ = 1/2!$

This discovery led to some interesting results, particularly using a six-fold symmetric purse fold.

Further development was made using a tucking movement. This allowed me to create a concave or convex vertex in a specific point of the grid obtained from the earlier folds (figure 3).

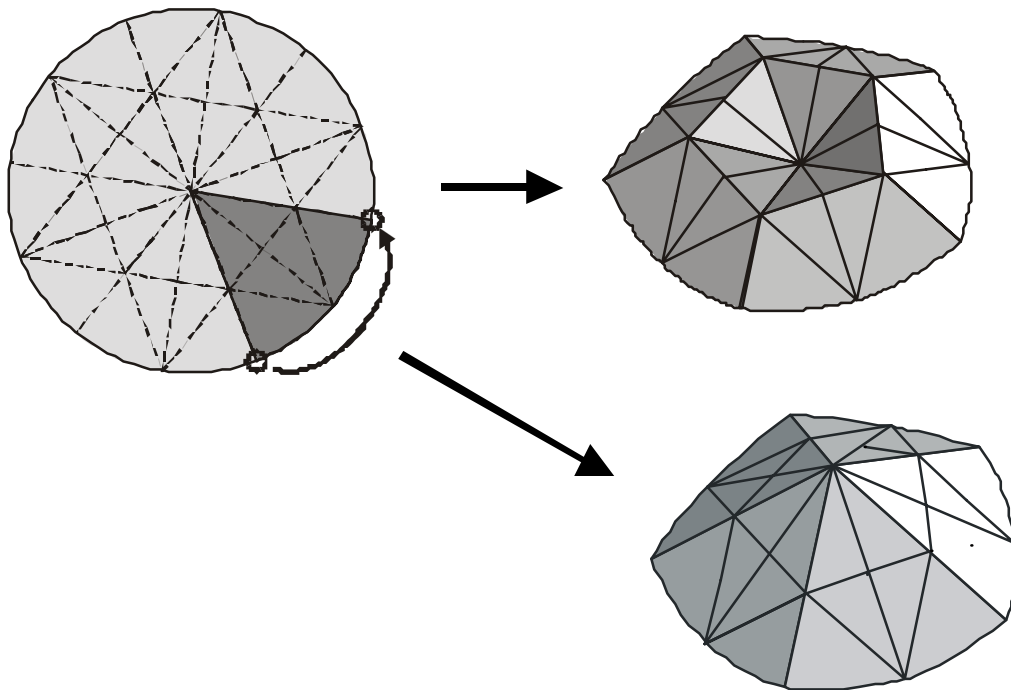


Figure 3: concave or convex vertex

What happens if you apply this tucking system repeatedly to your paper? Can you close up the shape? The answer here is yes, in many different ways. In the above drawing, 60 degrees were removed leaving 5 equilateral triangles exposed. It is also possible to remove 120 or even 180 degrees, or to leave a “vertex” of the grid unchanged. Figure 4 shows an example of a shape combining 6-vertices and 4-vertices. It can be made from a circle subdivided into 8 along the diameter.

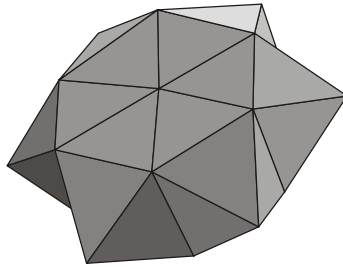


Figure 4: 6-6-4-6-6 Deltahedron

This folding method can of course yield all kinds of results, but in certain cases, a new problem emerged. Because you can only generate subdivisions of the diameter into powers of 2 (2, 4, 8, 16, etc.), if you need, for example, 10 or 12 rows, you will use a large circle and much of its surface will have to be tucked away inside the shape, mostly at the closing point. In the case of the regular 5-6-5 deltahedron, (Figure 5), you need 12 rows, leaving 2 extra rows at each end (for a total of 16). The paper that needs to be tucked away becomes so unwieldy that I decided to remove some of it through cutting. This is of course a departure from traditional Origami techniques, but I felt it justified by the interest of the result. The cutting technique itself is interesting because it mirrors the structure of the final shape (see also [2]: From circle to icosahedron). It consists of the removal of a wedge-shaped piece corresponding to 1/2 the angle that would be tucked away. This allows for the retention of a tab that will be used for assembly. In the case of the example of figure 5, the wedge has an angle of 30 degrees and the greyed out areas correspond to the tabs.

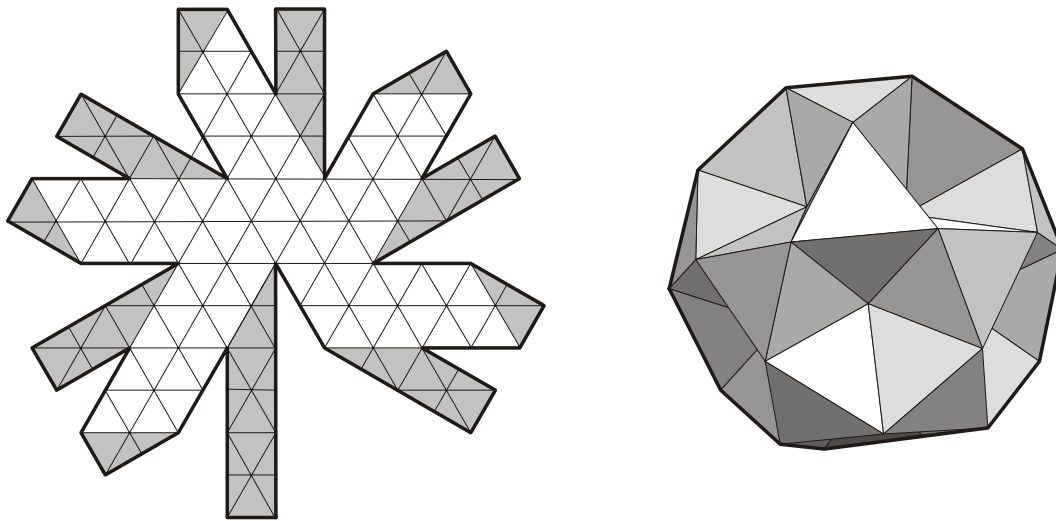


Figure 5: Geraldine (Endo-pentakis-icosi-dodecahedron)

The shape thus developed was so interesting in its own right that it became the centerpiece of my experimentations. Lacking the knowledge to determine its scientific name, I called it Geraldine (meanwhile John H. Conway was nice enough to identify it as an **endo-pentakis-icosi-dodecahedron**), and continued to experiment with it. The first step of this further study consisted in building a larger model using the same technique. Geraldine II was made out of a plastic sheeting used in buildings as a vapor barrier, shish-kebab sticks for rigidity, and hot glue. The finished shape was 1 meter (about 3 1/2 feet) in diameter (figure 6). The initial circle was drawn using a pencil and a rope, and three people assembled the whole shape in a private home during a single afternoon.

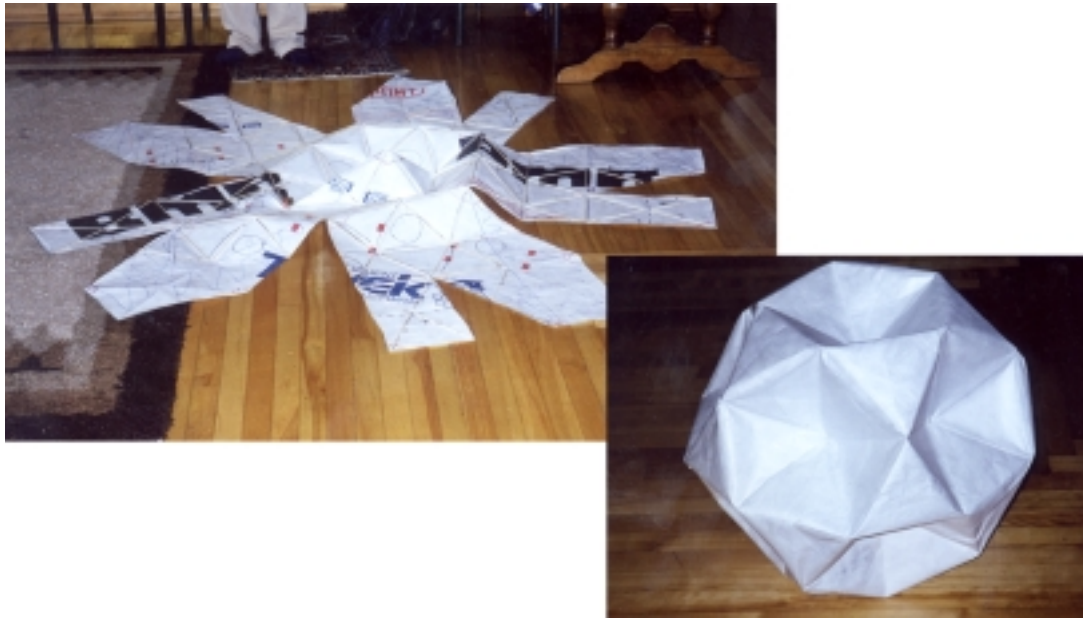


Figure 6: Building Geraldine II

Building the larger shape was an enriching experience, very different from the assembly of the small hand-held model. The material reacted differently, yielding much more to gravity, and we had to work together using our whole bodies in collaboration, making the whole event much more of an all-encompassing experience, convincing us that we needed to build Geraldine even bigger!

Geraldine III (figure 7) was made using a modular system based on kite technology, and the barn-raising event took place at Lanier Middle School (Houston, TX) in April 1999, and again at the Bridges: Mathematical connections in Art, Music and Science Conference (Winfield, KS) in July 1999[1]. The modular system, a further step away from Origami, had the added advantage of allowing us to build more shapes based on the same principle, including the shape of figure 4, flexagons, twisted tubes, etc.[3].



Figure 7: Building Geraldine III

Circular Origami has so far shown promising results. Even though with Geraldine I have made a significant departure, finding a complex enough field in its own right, I have continued to study circles and their folds. After all, its potential is probably as infinite as its symmetries! In the next section, I will discuss more of the mathematical aspects of my results.

Some of the mathematical results

It would be difficult to exaggerate the importance of the results obtained after the realisation that $\sin 30^\circ = 1/2$ opened the door to 6-fold symmetries in the folding of the circle. There are many more mathematical theories and theorems that can be illustrated through my developments in circular Origami.

Figure 3 begins an illustration of the Euler characteristic (faces - edges + vertices = 2) and the angle deficit theorem [4] according to which any closed polyhedron without holes in it (no donuts allowed) will have a total missing angle of two whole turns. Using the example of figure 4, one can see that there are only six vertices where material was tucked away (the six 4-vertices). Furthermore, at each vertex, two whole equilateral triangles (so 2×60 degrees) were removed, for a total of $6 \times 2(60) = 720$ degrees, two whole turns! In the case of Geraldine, there are twelve 5-vertices (in the middle of the dimples) and the rest are 6-vertices, so $12 \times 1(60) = 720$, again, two whole turns. But what happens when some of the vertices have *more* than six triangles meeting? Taking the example of the stella-octangula (which can also be built from a circle, albeit a little differently; figure 8). The shape comprises eight 3-vertices and six 8-vertices. The calculation would therefore be: $8 \times 3(60) - 6 \times 2(60) = 720$. In this case the second term in the calculation is negative because it is an angle increase when we are calculating the deficit.

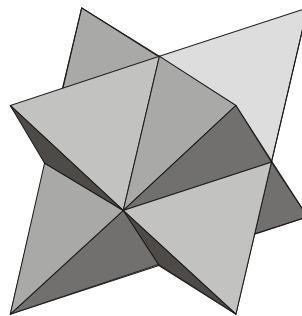


Figure 6: Stella-octangula

Returning to Geraldine, it was soon apparent that she had the same symmetry and general structure as the regular icosahedron (in fact, if you bump out all the dimples, you obtain an icosahedron of ‘edge-length’ equal to 2). It is also possible to build a ‘dimpled’ dodecahedron using the same system. This observation prompted me to ask the following questions: what is the generating principle of all these related shapes, and are there more similar ones. The second part of the question can be answered in the positive simply by observing that the ‘soccer ball’ can be similarly dimpled on its pentagonal faces to obtain, again, a polyhedron of the same family (made of only equilateral triangles and possessing the icosahedral symmetries). Trying to answer the first question took us, via vector geometry straight to number theory [5]. Using these varied mathematical tools, we were able to demonstrate that there is indeed a generating principle, and also that the shapes generated have some qualities that are constant. An example of this is that in the case of the icosahedral family, the shapes will always be composed of a multiple of 20 faces. Further, this multiple corresponds to the square of the shortest surface distance (always a whole number) between two ‘adjacent’ 5-vertices.

There are of course more observations of a mathematical nature to be made here, including such challenges as color distribution, whether the previous is applicable to other symmetry groups, etc. The most important aspect to note here is the constance of the parallels that can be made between folding and understanding mathematics. Having understood this, it was easy to envision a connection with mathematics education. Having worked with a doctoral student in mathematics at Rice University, I was able to transfer these experiments into a school setting, witnessing first-hand the pedagogical worth of the exercise.

Education

The education applications of the preceding experimentation began in 1999 through the Rice University School Math Project, a resource for mathematics teachers in the community in Houston, TX.

Early sessions took place at Lanier Middle School. The first session consisted of the building of an icosahedron from a circle using the paper-folding method referred to in the first part of the paper. After a few exercises aimed at practicing using the triangles, the second part of the event consisted of the building of Geraldine in one of the courtyards of the school. The two parts together were significant because they gave the students the opportunity not only to handle the shapes themselves, but also to build them in such a way that the process reflected the final object every step of the way. This is a definite departure from a typical exercise where the teacher gives the students a model of a polyhedral net that mysteriously assembles into the shape the teacher promised. It is not possible to overemphasize the importance of this aspect. If the students can follow the reasoning process from beginning to end, they never need to take the teacher's word for it. Furthermore, this type of exercise, particularly the part with 'life-sized' triangles goes a long way towards reconnecting the abstraction of mathematics with reality. Finally, the manipulatives used in these exercises are ideal for discovery learning as they let the students experiment, and more specifically they can experience the thinking process with their whole body, rather than only the finger tips in the case of a hand-held model or than the eyes only in the case of a film or computer program.

The circular Origami exercise and the large triangles have been used successfully at various levels from K-12 all the way to an undergraduate mathematics environment, both with advanced and at risk students. Work is set to continue in this direction, with a particular emphasis on documenting the various lessons as well as developing further applications.

Conclusion

The story of my experience with circular Origami is by no means ended at this point. The success of the applications both in mathematics and education demonstrates the potential for further development. All that can be said is: 'til next time!

[1] Knoll, Eva, Morgan, Simon, *Barn-Raising an Endo-Pentakis-Icosi-Dodecahedron*, Bridges Conference Proceedings, Winfield, Kansas, 1999

[2] Knoll, Eva, *From circle to icosahedron*, Bridges Conference Proceedings, Winfield, Kansas, 2000

[3] See the some more pictures at the ISAMA Website: <http://www.isama.org/>

[4] Morgan, Simon, Knoll, Eva, *Polyhedra, learning by building: design and use of a math-ed tool*, Bridges Conference Proceedings, Winfield, Kansas, 2000

[5] Knoll, Eva, *Decomposing Deltahedra*, ISAMA Conference Proceedings, Albany, NY, 2000