DISCUSSING THE CHALLENGE OF CATEGORISING MATHEMATICAL KNOWLEDGE IN MATHEMATICS RESEARCH SITUATIONS

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ABSTRACT

Starting with a quotation describing mathematical research, this paper presents ways of providing students with comparable experiences in mathematical research, in the classroom. The paper focuses on the benefits and implications for the students of such experiences. "Real mathematics research-situations" are defined, and the didactical goals of these situations, as they are experienced are elaborated on. These elements are presented through examples, looking at similar situations (research-situations) in two contexts and using different theoretical frameworks.

Key words: mathematical research situations, mathematical reasoning, experience of knowing, definition construction processes.

INTRODUCTION, THEORETICAL FRAMEWORK AND AIMS

It is widely accepted that school mathematics differs considerably, in scope as well as in purpose from what mathematicians do; the necessities of the classroom are certainly different from those of cutting edge scientific research. It is less obvious why the specific activities of students in the classroom are so different from what mathematicians do in their research. According to Corfield (2003, p. 35):

> Theorem proving, conjecturing and concept formation make up the three principal components of mathematical research. The brilliant observation of Lakatos […] was that these components are thoroughly interwoven. […] Mathematicians perform these activities simultaneously […]

In contrast, in the classroom, such three activities are marginalised to the extreme, so that students are rarely involved in research situations. We use naturally the word "research situations", but it has to be defined more precisely. We propose in this paper a characterisation of what we called "real mathematics research-situations". Of course, there are many such research-situations, and the ways we could approach them are also multiple. In this paper, we describe possible issues surrounding research situations by focusing on two sets of considerations: epistemological and social/didactical. These considerations stem from the following examinations:

Q1: why investigate research-situation?

Q2: what qualifies a research-situation as such?

Q3: what does it mean to "implement a research situation in the classroom" and what are the didactical goals? What can a student learn through research-situations?

In addition, to illustrate the implications of their implementation in the classroom, we call on two cases of such research-situations. These situations will be characterised in terms of the above-cited considerations as well as their individual contexts.

RESEARCH SITUATIONS: A MATHEMATICAL AND DIDACTICAL CHARACTERISATION

Mathematical (epistemological) characterisation

We call 'research-situations' situations of a mathematically open nature. This characteristic needs to be fulfilled for all parties engaged in the situation (students, teacher, professional researcher). In particular, such situations may be still open in the ongoing professional research. In addition, accessibility of such research-situations is established by insuring that the mathematical pre-requisites be of minor importance: anybody can engage in research-situations because it mobilizes only basic mathematical knowledge (of integers, or basic geometrical forms, etc.).

The situations are also characterised by a purposeful focusing on the engagement of students in a mathematical research process, and therefore, the emphasis is placed on the experience of process, as opposed to the acquisition of conceptual/technical 'content knowledge'.

An example of this can be found in the work of a 'Research Situations for the Classroom' (RSC) team called "Maths à Modeler" (http://mathsamodeler.net), centred on an initial presentation by a researcher for classes at different levels (from primary school to university). The study of these RSC suggests that they differ from what is traditionally known as problem solving by several characteristics:

- Questions are raised in a similar way to the approach of open conjectures in ongoing mathematical research and, sometimes, they are 'open' questions;
- the mathematical objects considered are not necessarily part of the explicit school curriculum, and the questions are generally not given in a mathematical form;
- there need not exist a unique answer (or any answer at all);
- a solved question can possibly lead to other new questions;
- the knowledge involved is more often 'transversal', such as arguing, conjecturing, proving, modelling, defining… bringing us back to Corfield's view.

This reflection about RSC is based on the notion that a researcher can, and often must, select his own suitable framework of resolution, modify the rules or redefine objects or questions. This is precisely the type of practice, fundamental to mathematical activity, in which we aim to involve students despite the fact that this type of practice is not frequent in class, and even seems practically taboo in many circumstances (Godot & Grenier, 2004).

The concept of 'transversal knowledge' cited in the list above has not yet been formally defined. At present, it refers to the skills and knowledge which straddle various mathematical domains and are used in a whole variety of mathematical contexts. In that respect, it relates to what Bruner defined as 'non-specific transfer or, more accurately, the transfer of principles and attitudes' (Bruner, 1960, p. 17). Indeed, transversal knowledge and skills allow the knower (student) to navigate within different mathematical domains. They are therefore more valuable as they are less context-bound. In this context, they include proving, conjecturing, refuting, creating, modelling, reasoning by induction or by decomposition/recomposition, extending but also transforming a questioning process, reasoning non-linearly, building definitions and having a scientific responsibility.

Didactical characterisation

It is essential, for these epistemological characteristics of the experience to be fulfilled, that the teacher takes a specific position, similar to that of a researcher faced with an open problem, and comprising an awareness of the involved transversal knowledge (Godot & Grenier, 2004). This brings us to the social and didactical contract (Brousseau, 1997) that will produce the appropriate context for these activities to lead to the desired goals. There are many ways to grasp and characterise a research process: a didactical viewpoint is proposed in Godot & Grenier (2004) for instance, and Rota, in his introduction to *The Mathematical Experience*, explains that:

> A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure our minds are not playing tricks. (Davis & Hersch, 1981, p. xviii)

The didactical contract needs to focus on the aspects of mathematics research which are relevant and the way these aspects can be made present in the experiment. To illustrate, Rota's description suggests that the creative process in mathematics research is a messy activity with no guarantee of successful results. This contrasts with the traditional classroom experience where the students seek a definite solution that the teacher already knows. Each aspect such as this one needs to be evaluated for usefulness, then, if possible, reformulated for the classroom context.

Many of the teacher's decisions impact on the format and content of the activities. Firstly, mathematical research is a creative endeavour, and cannot be easily framed into the occasional one-hour session. The timeline must therefore be made to reflect this characteristic: the student must have the opportunity to appropriate the experience of the process reflexively, and to pass the frustration point where the temptation to give up is the strongest.

Secondly, as discussed above, the mathematical concepts have to be accessible enough to the participants that they can focus on their research process (in a reflexive endeavour). This is designed to ensure that the responsibility and power is shifted to the participating students and it is accomplished in three ways. To begin with, the students are given ownership of the experience through their own process of formulation of the question/problem. In consequence, and secondly, the teacher does not know the answer in advance, or even if there is one. Thirdly, and most importantly, the teacher works as a sounding board only, in order to avoid leading the process.

Finally, these constraints need to be recombined with the needs of the curriculum, including the teaching of basic notional mathematics. The following two sections illustrate instances where this plan was implemented.

EXAMPLE 1: RESEARCH SITUATION AND EXPERIENCE OF KNOWING

In 2003, a class of elementary student teachers in an American university spent two months in one of their required mathematics courses on a research project 'at their own level' (Knoll et al., 2004). They spent a month investigating research situations similar to the one described above, in informal groups. In many cases, there was not even a specific question, let alone a unique answer. In the second month, the students chose one of the investigations and took it or one deriving from it to a deeper level. Remember, these are not subject specialists; despite that, each student or group of students conceived their own topic!

In this particular case, the students investigated geometry topics such as proper colourings[1], polyhedra, and tilings. Note again, that the mathematical objects were easily accessible. And of course these problems, though seemingly trivial to a mathematician, have not all actually been solved; their resolution would not increase the canon, because it does not require the development of new mathematical tools, and so they are left out, but that is another story.

Concerning the anticipated achievement of the participating students, the study focused on their relationship to the subject of mathematics, including attitudes, beliefs and practices. Indeed, if a knower sees mathematics as made of a closely interconnected network of concepts, skills and relationships, she will be more likely to operate at a higher level than if she regards it as an amalgam of disconnected facts and procedures. To illustrate, the theoretical framework is summarised into a table with the action of knowing ('modes of knowing') in one direction, and the object of the knowing ('notions') in the other (see Table 1, below).

In this model, both categories are divided further, creating a matrix describing various situations. The key to this categorisation is that *different people could place experiences of knowing the same mathematics in different cells*.

In addition, the columns are distinguished by whether their content is (a) reproducible, (b) transferable, or (c) reconstructible. Clearly, a notion is not known if

[1] A proper colouring is a colouring of a subdivided system such that no adjacent cells share the same colour.

it cannot be reproduced. Further, if you know *why* something is the case, chances are you would be able to use it in a different situation. For example, understanding that the 'carry the 1' action in two column addition comes from the notion of place value, can lead to being able to transfer this to the case of two column multiplication without being told.

Modes of knowing **Notions**	**Knowing that/how** *Reproducible but neither transferable nor reconstructible*	**Knowing why** *Reproducible and transferable but not reconstructible*	**Knowing when** *Reproducible and transferable AND reconstructible*
Convention arbitrarily chosen	Memorised information and use	Nothing to understand/derive	Cannot be reconstructed by reasoning
Application moving from theory to practice	Subjectively the same as a 'convention': memorised information and use	Derived from other notions using the logical structure of mathematics	Derived from other notions using the logical structure of mathematics
Theorisation moving from practice to theory	Subjectively the same as a 'convention': memorised information and use	Derived from other notions using logical structure of mathematics	Derived from other notions using logical structure of mathematics

Table 1: Ways of Knowing and Notions

In the case of 'reconstructible' knowledge, the knower has grasped the mathematical structure underlying the notion to such an extent that, given the need, she would be *able to reconstruct it*. This distinction is important in that it implies a deeper understanding of the mathematical structures from which the specific emerges, giving it more transferability potential and a more wide-ranging applicability, making it more fundamental and going back to the notion of transversal knowledge. This is not saying, however, that it applies only to higher domains of mathematics.

Let us now look at the rows, the categories of knowledge. There are many models for this in the literature on mathematics education (Piaget, 1970; Bell et al. 1983; Skemp, 1987, Hejný, 2003; etc.). They are generally constructed to emphasise one aspect or another, or to make key distinctions. For the sake of clarity, in the present case we will refer to the fragments of specific knowledge as notions, avoiding thus the need to specify to whose definition of 'concept', 'skill', etc. we are referring.

The first and perhaps most important distinction separates a convention from the others. This distinction is important in that it is carried through to the ways of knowing. As can be seen in the table, a 'convention' is not the result of a logical derivation from a more basic or fundamental entity. It is somewhat arbitrary. This is key: Considering a given notion as a convention is a kind of fallback position, when the learner just cannot grasp something. The learner will then regard the notion as something that was decided for reasons that remain obscure, or even arbitrarily determined by someone else and take it at face value. This mechanism can be the correct one, but mostly it will lead to problems.

The second distinction is between applications and theorisations. Both categories contain the results of mathematical reasoning, unlike conventions. In addition, the two categories are distinguished through the direction of activities that call on them. In the case of application, notions are used to solve problems, to execute algorithms, and perform other activities that take the knower from the general, abstract, theoretical, to the specific, concrete, applied, as in the majority of traditional classroom work.

In contrast, the theorisation category operates from the specific, concrete, applied case to the general, abstract, theoretical. This is what is used in the mathematical activities described earlier: proving, conjecturing, refuting, defining, etc.

Evidently, the right end and the bottom of the table represent deeper thinking. In addition, the whole network is interconnected. Unfortunately, in many models, the distinction between the centre and right columns is left out and the two are collapsed or even left out altogether. This deeper thinking, which relates to the transversal knowledge described above, is therefore little emphasised, *or verified and assessed* in conventional classroom activities, even though it is much more fundamental and most importantly more resilient.

The important point to consider in this model is that moving towards the right *does not imply* delving into higher mathematics. The level of mathematics constitutes a third, independent dimension, and theorisation notions can be accessed in the context of very accessible mathematics, as indeed can ‘knowing when’ be achieved.

The project formed an experiment focusing on this. In fact, the course was designed to direct the students’ attention onto their engagement in a mathematical research process, as discussed earlier, with special attention to the elements on the right and lower ends of the table. This was done using a whole battery of strategies, including the use of writing and portfolios for assessment, reflections and other interactions with the didactician. This last in particular was encouraged through the use of reflective student journals and the emphasis of the final project report on the students’ process as opposed to their results. The participants found generally that the atmosphere of the classroom was unlike the mathematics context they were used to. Several commented that their outlook if not their feelings had changed, and many realised that mathematics was more than they had previously been led to believe (Knoll et al., 2004).

EXAMPLE 2: RESEARCH SITUATION AND DEFINITION CONSTRUCTION PROCESSES

Processes of defining represent an important part of mathematical activity, as underlined by Lakatos with an example concerning the immersion of a proof in a classification task:

> *there are other ways of communicating meaning than definitions. I, for one, shall initiate my pupils into the problem-situation which I am dealing with not by definitions, but by showing them a cube, an octahedron and showing that for these V-E+F=2. Then I shall ask for the domain of validity of this formula. (Lakatos, 1961, p.69)*

In this context, Lakatos shows that a definition is not only a tool for communicating, but also a mathematical process taking part in the formation of concepts. In the example at hand, the aim consists of a characterisation of markers in order to examine the concept formation process, and, in particular, to identify specific statements in the defining processes in order to say something about concept formation.

If we consider classification tasks as a part of the definition building context, this definition building process itself takes place within the wider problem that is the search of a proof, which in turn catalyses the construction of the concept, of polyhedra for example (Lakatos, 1961). If we concentrate our attention on a classification situation as a definition construction situation (it can be a particular RSC), this may appear simple: one takes some examples and counter-examples of a mathematical object and asks for a definition. We have experimented with this type of defining situation with the mathematical object 'tree' (see Ouvrier-Buffet, 2003) and also with the mathematical object of the 'discrete straight line' (see Ouvrier-Buffet, 2004). Both these situations have been conducted with students in their first year of university (scientific and not scientific sections). These mathematical objects are noteworthy because they are accessible by their representations, and are non-institutionalised, thus no pre-existing definition of these concepts exists. Furthermore, let us notice that to classify is a familiar task, both in everyday life and in sciences (in geometry or in biology for instance). Moreover, some students' conceptions about mathematical definitions (what definitions are or should be, what they do, the aspect they should have, etc.) will certainly have a leading role in such a situation, and may represent an obstacle to the defining process, or even a catalyst. We have to be aware of this fact, but the main question stays: are students capable of awareness of their own defining process? We want to study this kind of reflexive process. So, the challenge is now to characterize defining processes and situations in which a definition has to be built.

There exists a model for defining processes (Ouvrier-Buffet, 2003) which emphasises the *operators* and *controls* taking part in the processes. Modelling defining processes involves exploring the procedures implicated in the creativity of professional research mathematicians when they build new concepts (admittedly this is no small challenge!

That is why the roots of the presently characterised operators and controls, which are taking part in a defining process, are epistemological and philosophical[2])

The study also involves the identification of the markers of these processes in order to analyse how students define. These markers also allow a first characterisation of some key tools at the teacher's disposal for a definition building activity. Let us present some element of this. Remember that the teacher, in an RSC, is *not* the holder of knowledge. He has to adopt the position of researcher, like the students. Even then, there are 'didactical levers' such as recalling the instructions and asking for a definition.

The study of the concept of definition shows that a defining process is based on four poles, graspable in three epistemological conceptions: one concerns the *construction of a theory* (Popper, 1963), another deals with *Problem-Situation* (Lakatos, 1961), and two other poles concern the *logical* and the *linguistic* aspects (Aristotle), respectively. In this context, the teacher (who becomes a **M**anager-**O**bserver in an RSC) may interfere in the defining process of the students through logical requests, linguistic or axiomatic exigencies or the supply of given counter-examples. The MO can also ask for the construction and/or recognition of an object (tree or discrete straight line in a classification task for instance): it is a request obviously related to the function of the definition (does the definition help to recognise or construct the discrete object?). For instance, a question like this: "draw a discrete straight line crossing these two given pixels" engages students in a new reflection, of an axiomatic kind; the uniqueness of such straight lines is of crucial importance, and implies thus a new movement in the defining process.

This last statement necessitates a wider characterisation of the teacher's levers for defining situations. It can be comfortable for the teacher, because this kind of activity (the didactical contract for students is to build a definition) leads to a product, re-usable in a course. The teacher also remains in control of the process to a large extent, because there are clear goal posts.

CONCLUSION/ OPENING REMARKS

This paper brings an overall picture of the potentialities of research situations for the classroom. RSC give us an opportunity to work on scientific processes, constituted by students' experiments with different cognitive attitudes: doubting, conjecturing, refuting (generating new counter-examples), testing etc. In particular, the processes of defining can be modelled through four main items: formulating, logic, heuristic,

[2] We will propose an integrated picture of these operators and controls, in relation with different kinds of defining situations during the conference, with a poster entitled "On Modelling Conceptions about Mathematical Definitions". We will also present an illustration of the use of this model with a definition-construction situation.

theorising. The situations also give potential for students' reflections on wider issues, for example about the nature of mathematics, or even knowledge in general.

The discussions initiated in this paper illustrate the challenges facing education researchers interested in the research-situations as applied in the classroom. Further work concerning the nature of transversal knowledge and the understanding of a mathematical concept is fundamental. We have now to continue the implementation of research situations in the classroom in order to refine the characterisation of transversal knowledge and to define "properly" the learning in a specific context: that of the creation of knowledge, both in the discipline in general and in the mind of the learners.

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