Building a Möbius Bracelet Using Safety Pins: A Problem of Modular Arithmetic and Staggered Positions

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Abstract

This article reports on the resolution of a mathematical problem that emerged when two ideas were brought together. The first idea consists of a method for constructing a decorated bracelet made with safety pins that are strung together at both ends, creating a band. The other is suggested by the word *band*: why not introduce a twist and make the bracelet a *Möbius* band? As Isaksen and Petrofsky demonstrated in their paper [1] discussing the knitting of a Möbius band, the endless nature of the band's single face and edge introduces an additional design constraint, particularly if the connection is to appear seamless. To make the creation appear seamless, the decoration applied to the design must itself be regular, as this helps the eye travel along the endless length. The paper discusses the mathematical and practical constraints of this result for a design that uses a repeating pattern throughout the band, first in the standard design, then in the Möbius bracelet. This resolution involves some simple modular arithmetic and an unusual way to lay out the pins in preparation for their being strung together.

1. Introduction

Several years ago, I came across an interesting design for a bracelet; it is made of a large number of safety pins, on which beads have been threaded, and which are then strung together at either ends, resulting in the bracelet shown in Figure 1, below. This design is very interesting in that it makes use of easily accessible material while producing a rather massive and decorative piece of jewellery.

Figure 1: *The 'standard' bracelet made of safety pins strung together*

The construction of the bracelet is straight forward. The first step is to string beads onto approximately 80 safety pins. Because commercially available safety pins have a spring (a 540^{o} twist) at their tail end, beads can only be strung on one side. Figure 2, left, shows two pins holding beads that will produce different designs. The pins are then strung together in an alternating pattern so that one design is visible on each surface, and so that heads and tails alternate in each edge. In Figure 2, right, I strung the two different pins together showing the beginning of the stringing process.

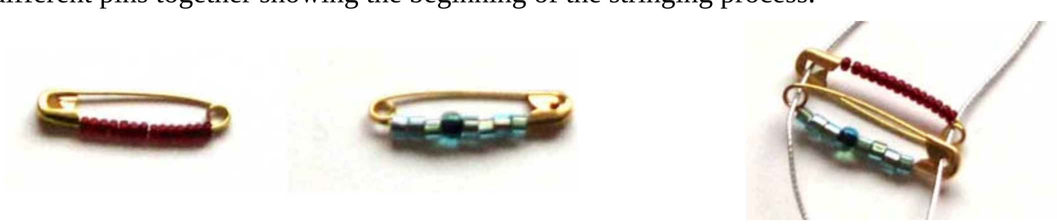

Figure 2: *Stringing the beads, then the pins*

When all the pins are on the string, the ends of each string are tied together, most securely using a granny knot, and a closed band is created. In this standard design, the finished bracelet has two 'faces', an inside one and an outside one; it can be turned inside out and therefore can be worn two ways: a single bracelet can carry two designs. The pins in Figure 2 have different bead combinations, which will produce two designs. Figure 3, below, shows the finished bracelet, inside-in (left side), and inside-out (right side).

Figure 3: *Two ways to wear a finished bracelet*

2. Analysing the construction of the standard bracelet

The standard bracelet is constructed by pulling two elastic strings through the tails and heads of the lined up pins, after which the two ends of each individual string are tied together. The relative orientation of each successive pin being strung is an important component of the design. The pins can be strung together in several different ways. For example, all the pins can be in the same orientation, and they can simply be translations of each other along the bracelet. The specific dimensions of each part of the beaded pin, however, will then be multiplied, and if there is a difference between these, it will be magnified. The side that has beads on it, to start, is now thicker than the other, and stringing all the pins the same way, with the beads facing the same therefore means that one 'face' of the band is somewhat bigger than the other, and in this method, the band will curl up on itself. It would be an improvement to alternate the position of the beaded side, up, then down, etc. so as to eliminate this tendency. A consequence of this method is that both 'faces' of the band now show beads, and therefore, the bracelet can hold two different designs. The two pictures in Figure 3 show the same bracelet worn so as to show each of the two designs.

In addition to the thickness of the beaded side of the pins, a consideration has to be made because the head of the pin (where the closure is located) is often thicker than the tail (where the 'spring' is located). If all the heads are on the same edge of the finished band, that edge will be longer than the edge containing the tails, and the band will want to trace an arc away from the heads. Alternating the beaded side, up-down, is therefore not sufficient to ensure a straight band design: the heads and tails also need to alternate.

For the band to be as balanced as possible, the pins therefore need to be rotated, relative to each other, by 180^o, around an axis that is parallel the band, in the middle of it, that is, perpendicular to the plane within which the pin rests. In Figure 2, right, the two pins are strung so that the beaded and unbeaded sides are alternating and so are the heads and tails. The alternation of the pins in these two ways has several consequences. Firstly, it means that there are actually two groups of pins that are strung, alternatively, but otherwise independently of each other: the two faces of the bracelet can have completely different designs on them (in the example in point, one face only has red beads, and the other shows a sea-green wave pattern, which I discuss more extensively in a later section).

As the two sets are alternating, in order to preserve the balance of the two sides, and show seamless, continuous surfaces, the number of pins in each of the two sets should correspond. If the first pin is strung bead-up, then all odd-numbered pins are bead-up and all even-numbered pins are strung bead-down. At the 'seam', the last pin should be strung bead-down, so that it alternates properly with the next pin, which is #1, therefore bead-up. This means that the total number of pins is necessarily *even:* for every pin that is bead-up, there is a corresponding one that is bead-down.

This design makes extensive use of parity: there are two edges (and therefore two elastic strings), two faces that can show two different designs, and the construction requires an even number of pins. In designs that show such a high identification with parity, a question can be posed concerning the handedness of the finished product. In this particular case, if we consider the pins as being symmetrical across the plane within which they rest, and if the decorative beads produce a pattern that is symmetrical along the band, then the overall design is not handed. If the design itself were handed, as shown in the later part of the paper, the bracelet would be handed.

3. Introducing a coloured wave pattern

In the previous section, it was determined that in order to have a seamless, continuous band that does not show a starting point, there must be an even number of pins, $2n$. The next step is to determine whether a decorative pattern can be integrated in a way that will still preserve this seamless quality. In the case of the proposed sea-green wave pattern, each pin supports a series of seven small light-coloured beads, and one larger dark bead. The position of this bead changes, on each successive pin, from 2^{nd} to 7^{th}, and back again: 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, etc. Each cycle of the pattern therefore requires 10 pins, as shown in Figure 4, below.

Figure 4: The ten pins of a single cycle

Figure 5: The eleven pins in the modified single cycle

If this pattern is to be applied on a seamless bracelet, the number of pins showing their beads at the same time needs to be a multiple of 10, otherwise a cycle will be incomplete. Simultaneously, the alternate group of pins has to total the same number. This works well with an 80-pin design, where the cycle would be used four times, using 40 pins, and the cycle on the opposite face (in this case identical pins with red beads) would use another 40 pins.

When I built the bracelet, I soon realised that it would be too tight as it was designed. On the other hand, I did not intend to add 20 pins (10 bead-up and 10 bead-down), to extend it, as this would make it too large. Instead, I modified the pattern slightly, as shown in Figure 5, below: the crest of the wave is composed of two identical pins, whereas the trough remains the same (I extended the display to show the continuity).

This creates a pattern of 11 pins, in four complete cycles, for a total of 88 pins, counting both bead-up and bead-down groups. The fact that the number of pins in a single cycle is odd is not a problem because the total number of pins in each group (up or down) need not be even; only the total needs to be even, which it is, with $44 + 44 = 88$ pins (the total is and must be even because for every bead-up pin there is one that is bead-down). In the photo on the left of Figure 3, above, the middle area of the bracelet shows the ‘stutter’, where two identical pins follow each other. Figures 6 on the next page shows the whole of each group of pins, before the bracelet is closed up.

Note that in the sea-green design (right of figure 6), the first pin on the left has the dark bead in position 5 (from the bottom up as the pin is head-down), and, so that seamlessness is preserved, the last one on the right therefore has the dark bead in position 6 (again, head-down). Also, as the pins with the sea-green beads are depicted head-down, the stutter is now in the trough. In the case of this example, the pins used are 1” brass safety pins, and the beads are coloured glass.

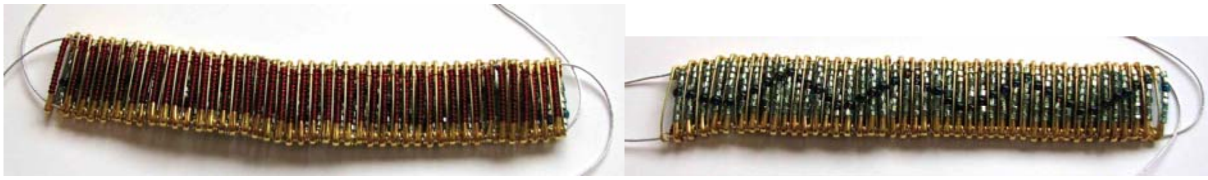

Figure 6: *The two sides of the standard bracelet before closing up*

The standard bracelet design can of course be assembled in irregular ways, using pins with randomly coloured beads, without regard for any patterns or a continuous finish, but the added constraint of seamless regularity brings out some of the mathematical structures of the design. Figure 7, below, shows part of both sides of the bracelet.

Figure 7: *Showing both faces of the bracelet, before closing up*

4. Modifying the design

Möbius bands are a well known mathematical phenomenon that has interested mathematicians and artist alike, including M. C. Escher, Max Bill, and many others. The simplest way to make a Möbius band is as follows:

> Take a long, thin rectangle of paper, and join the narrow ends after giving the strip a half twist [on one end]. [3; see also 4]

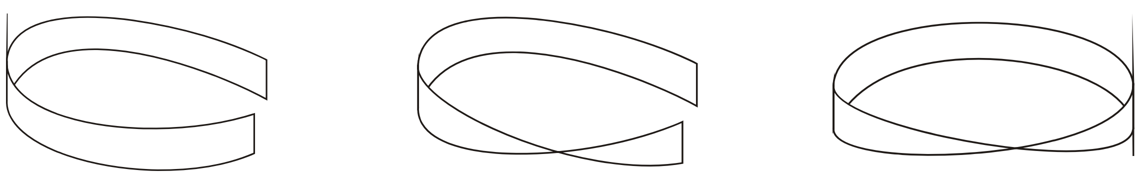

Figure 8: *A line drawing of how to construct a Möbius strip and the result*

The resulting form has interesting properties, including the fact that it only has one edge and one face [2]. To test this, take a pencil and draw a line along the middle of the band, continuing until you meet the starting point. Notice that you have passed the starting point, on the other 'side' of the paper, somewhere along the way, without crossing the edge at any point. Similarly, if you mark your starting point and follow an *edge* along until you return to the starting point, you will have touched 'all edges', without crossing the face, thereby tracing a single edge.

It is possible to implement the safety pin bracelet idea described above on the Möbius band structure, though allowances for this combination need to be made. Firstly, as we have seen, the Möbius band only has one edge: we will therefore only use one elastic string, which will somehow pass through each pin at both ends.

Just as the Möbius band's single edge means that there will only be one elastic string going around twice, continuously, the fact that there is only one face means that the finished Möbius bracelet can only have a single design, which will be continuous through *all the safety pins*. In addition, in the case of the standard bracelet, the total number of safety pins is necessarily even so as to conceal the starting (and end) point of the stringing. In effect, because the first pin counts as 'odd' (1 is odd), the last needs to be 'even', so that the bead-up, bead-down alternation was continuous across the seam.

In the case of the Möbius band, the twist that is given to one end before connecting impacts the number of pins required, and the situation therefore changes. If we start again with the first pin bead-up, then the last pin, *after the twist*, is required to be bead-down to ensure seamless continuity. Therefore, it needs to be bead-up *before the twist.* The pins that are bead-up before the twist are the odd-numbered ones, and so the last pin needs to be odd-numbered, and the total number is also odd.

These last two conditions combine into an additional one: the single, continuous design, combined with the requirement for an odd total number of pins, means that the pattern needs to fit onto this odd total number of pins. In consequence, and because the only way to produce an odd number through multiplication is to use two odd numbers, the cycle itself needs to be designed to fit an odd number, and there needs to be an odd number of cycles.

In the finished standard bracelet described earlier, the cycle was 11 pins long, and there were four cycles on each face, for a total of 88 pins (counting both the bead-up and bead-down sets). This does not work with the Möbius band bracelet. As the band is continuous across all the pins, it is possible to make 7 cycles of 11 pins (for a total of 77 pins) or 9 cycles of 11 pins (for 99 pins). In the first case, the 11 pins that are removed mean that the bracelet is too small to be worn. The 99 pins are a possibility, as the twist itself makes for a tighter final cuff size. In the case of the Möbius band I use to illustrate this paper, I applied a different pattern, with a 5-pin cycle, as shown in Figure 9, below.

Figure 9: *The 5-pin cycle used in the Möbius band bracelet*

The beads themselves cycle from black to white through various shades of blue, and, on each pin, they alternate with smaller silver beads. In Figure 9, the first pin on the left has a medium-blue, light-blue, white and black bead, and it is missing the dark-blue bead. The next pin showing its beads is missing the black bead, and has the dark-blue one. With a cycle of 5 pins, I chose to build the bracelet using 19 cycles, for a total of 95 pins.

5. Building the bracelet

In the standard bracelet described previously, as the patterns on the two faces were independent of each other, stringing was straight forward: I simply alternated the pins both in orientation and in their bead-pattern, selecting a red pin and stringing it bead-down, then selecting a sea-green pin and stringing it bead-up, careful only to ensure the proper cycle of the sea-green pattern. In the case of the Möbius bracelet, the stringing itself is trickier. In effect, I started stringing at two different positions in the whole pattern, because I simultaneously construct both 'faces', that is, two different parts of the single face.

The question, then, concerns the proper sequencing of the pins. The pattern shown in Figure 9 has five different pins, which we can call A, B, C, D and E. As I alternatively string bead-up and bead-down pins, the cycle A/B/C/D/E/A/B/C/D/E/etc. is stretched out, and in between each pair, a rotated pin (bead-down) is inserted. The bead-down pins follow the cycle in the same order, i.e. (if we use the – sign to denote bead-down), –A/–B/–C/–D/–E/–A/–B/–C/–D/–E/-etc.

There are several ways in which the two cycles can be combined. For example, I could simply follow each bead-up pin by the corresponding bead-down pin: A/–A/B/–B/C/–C/D/–D/E/–E/etc. There are other possibilities. –A can be between B and C, C and D, D and E or E and A. The difficulty is that this cannot easily be changed after the fact, without undoing all the stringing. The relative position of all the pins therefore needs to be determined in advance. In the diagram below, the two sets are positioned relative to each other:

… C D E A B C D E A B C D …
→… –A –B –C –D –E –A –B –C –D –E –A …←

Figure 10: *Thinking through the relative position of the alternating bead-up and bead-down pins*

A useful way to think about this problem is to consider each cycle of five pins (A/B/C/D/E) to be a single 'super-pin' that has a mid-point at the centre of the C pin. We know that there is an odd total number of 'super-pins' (there are 19), therefore, as they pass each other on opposite sides of the band, their centres also alternate, that is, the centres of the bead-down 'super-pins' are equally distant from the centres of both their neighbouring bead-up 'super-pins'. Therefore, the –C pin should be equally far from the two C pins that are on either side of it. In Figure 10, the –C pin is between E and A, and therefore 2 ½ pins from either C pins. To verify the accuracy of this solution, and to prepare for the stringing by laying out all the pins, I use the configuration shown in Figure 11, below (Figure 10 shows a short run of the 5-pin cycle strung together).

Figure 11: *Laying out the pins for the Möbius band bracelet*

The layout is useful because it integrates the Möbius band property: at the left of the circle, the pins fan out from the centre, whereas at the right, they are staggered and lie vertically. The twist is therefore integrated into the layout and the cycle is continuous throughout. In Figure 12, below, I show all the pins strung on one side, before I begin stringing the other side. Note on the left the first pin is bead-up and has a white bead near the tail. The second-to-last pin, on the right (which will be its neighbour, bead-up, once the band is twisted), shows a white bead in the second position from the tail. Conversely, the second pin on the left, which is now bead-down, shows the white bead in the third position from the tail, and its soon-to-be-neighbour, from the right side (the last pin), show the white bead in the last position before the head.

Figure 12: *Having strung one 'side' of the bracelet*

In Figure 13, below, the finished Möbius bracelet is shown, with the twist in the foreground. All the visible parts of the pattern are seamless and continuous, and this is the case overall. The Möbius bracelet design requires an odd number of pins, and if a pattern is to be applied seamlessly and continuously, it must consist of an odd number of iterations of cycles with an odd number of pins.

Figure 13: *The finished Möbius band bracelet*

6. Techniques and tips

The previous sections describe the mathematical processes involved in designing a Möbius band bracelet made of beaded safety pins. During the execution of the piece depicted in Figure 13, several techniques and tips were developed that can help the reader who wants to make her/his own bracelet. Firstly, the pins can on occasion have holes in the head which are too small for stringing. If this is the case, I recommend the use of a burnishing tool, as shown in Figure 14, below.

Figure 14: *Expanding the hole in the head of the pin*

Another useful technological tool for constructing the Möbius bracelet (and the standard one) is the use of 'Memory wire'. This metal wire is spun into slinky-like spirals that remember their curvature. The wire is circular in cross-section and comes in several sizes. It is typically used to make bangles and necklaces. The bracelet shown in Figure 13, as well as the piece shown in Figure 15 are made using this wire.

The closing of the bracelet using elastic string is best done using a granny knot, and with the memory wire, it is easiest to twist the wire with pliers so that the hook is inside the band. For extra stability, the wires can overlap within the holes for several pins before the ends are twisted.

7. Conclusion

The construction of the Möbius bracelet using the safety pin design launched an unexpected investigation into the internal structure of a Möbius band, using themes of number theory (modular arithmetic and parity) as well as topology (handedness). A final example of possible results is demonstrated by the following: In the South Bank of London, near Waterloo Bridge, there is a sculpture, shown in [5], which depicts a form similar to a Möbius ring that has a triangular cross-section. The idea would be to design a bracelet that has a triangular cross-section and a twist, so that the resulting shape has one face, which goes around three times. Using safety pins to connect the ‘faces’ as they pass each other, in pairs, results in the 3-Möbius bracelet depicted in Figure 15, below:

Figure 15: *The 3-Möbius bracelet*

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