Decomposing Deltahedra

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Abstract

Deltahedra are polyhedra with all equilateral triangular faces of the same size. We consider a class of we will call 'regular' deltahedra which possess the icosahedral rotational symmetry group and have either six or five triangles meeting at each vertex. Some, but not all of this class can be generated using operations of subdivision, stellation and truncation on the platonic solids. We develop a method of generating and classifying all deltahedra in this class using the idea of a generating vector on a triangular grid that is made into the net of the deltahedron.

We observed and proved a geometric property of the length of these generating vectors and the surface area of the corresponding deltahedra. A consequence of this is that all deltahedra in our class have an integer multiple of 20 faces, starting with the icosahedron which has the minimum of 20 faces.

Introduction

The Japanese art of paper folding traditionally uses square or sometimes rectangular paper. The geometric styles such as modular Origami [4] reflect that paradigm in that the folds are determined by the geometry of the paper (the diagonals and bisectors of existing angles and lines). Using circular paper creates a completely different design structure. The fact that chords of radial length subdivide the circumference exactly 6 times allows the use of a 60 degree grid system [5]. This makes circular Origami a great tool to experiment with deltahedra (Deltahedra are polyhedra bound by equilateral triangles exclusively [3], [8]).

After the barn-raising of an endo-pentakis icosi-dodecahedron (an 80 faced regular deltahedron) [Knoll & Morgan, 1999], an investigation of related deltahedra ensued. Although there are infinitely many deltahedra, beginning with the 8 convex ones [8][9], the scope of this paper is restricted to a specific class. First of all, the shapes under consideration have exclusively vertices with 5 or 6 triangles meeting. Since all the triangles are equilateral, the vertices all have 360^o or 300^o total flat angle. This means that there are always twelve 5-vertices, since the sum of the angle deficit (360^o-flat angle) at all the vertices of a genus 0 polyhedron always equals 720^o [6]). Second, the deltahedra have to be regular in the sense that each 5-vertex has a local 5-fold rotational symmetry that extends to the whole shape. We know from the symmetries of the icosahedron [7: p101] and [2], that the twelve 5-vertices are evenly spaced.

The intuitive route

Figure 1 shows examples of simple 3-D transformations applied to the icosahedron and the dodecahedron (the 2 platonic solids possessing 5-fold rotational symmetry). Using these transformations of truncation and dimpling (see figure), we found 4 deltahedra satisfying the above requirements:

- The icosahedron (1A) is one of the platonic solids.
- The endo-pentakis dodecahedron (1B) is transformed from the dodecahedron by 'dimpling' all the pentagons (endo-pentakis) .

• The endo-pentakis icosi-dodecahedron (1C) is transformed, either from the icosahedron or the dodecahedron, through first a truncation, then a 'dimpling'.

• Finally, the hexakis endo-pentakis truncated icosahedron (1D) is transformed from the icosahedron by first truncating it at the 1/3 point nearest to the vertex along the edge, then 'dimpling' the resulting pentagons[1].

But how does the endo-pentakis snub dodecahedron (1E) fit in? Can we find a sequence of simple transformations that will generate it from a platonic solid?

The platonic solids have both reflective and rotational symmetry. The simple transformations we used in figure 1 all preserve these symmetries. Due to its handedness, the endo-pentakis snub dodecahedron has only rotational symmetry. Therefore, this type of simple symmetry-preserving transformation cannot generate it from a platonic solid.

This defect prompted the search for a more systematic approach. In order to develop this approach, we need to find key structural properties of the deltahedra that can help us classify them. An exhaustive classification will help us develop a reliable recipe to generate the deltahedra of this class.

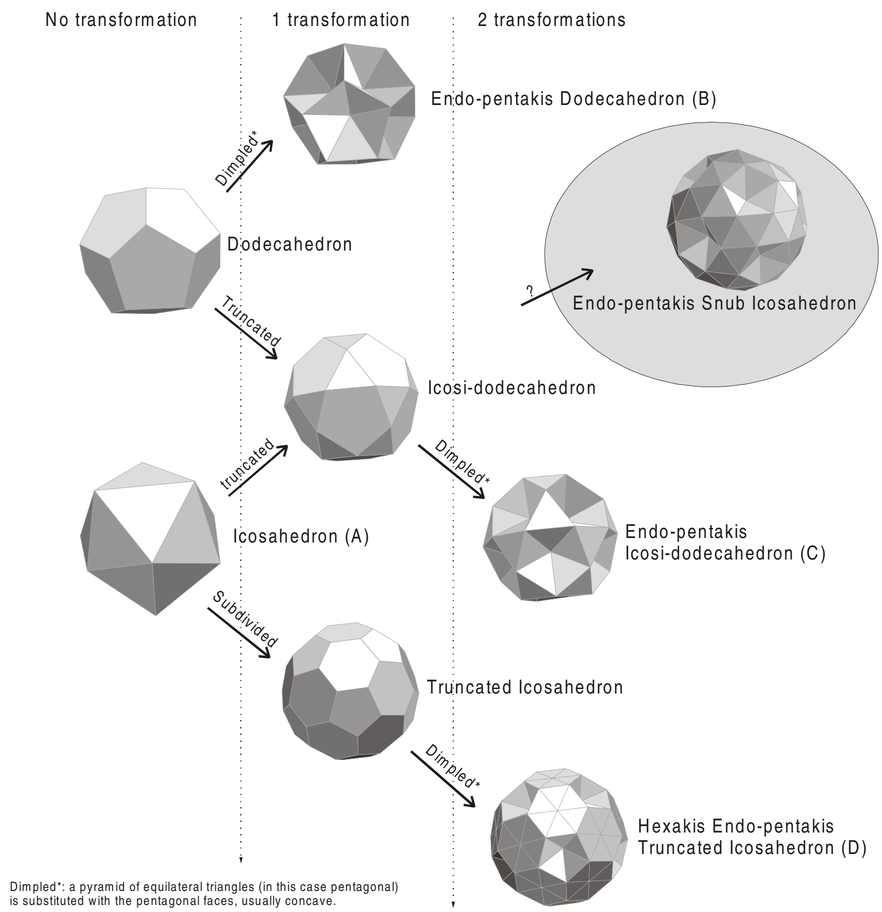

Figure 1: Deltahedra generated by simple 3-D transformations

[1] Note that the hexagons have also been subdivided into triangles to retain the deltahedral identity of the polygon, but that the ensuing triangles remain coplanar.

The Systematic Approach

From 3d deltahedron to 2d vector

Based on the snowflake-net construction method used in the barn-raising [6: 133-135], we focussed on the shortest surface path between two nearest 5-vertices on the surface of the deltahedra. A first look showed a similarity of structure between the three first deltahedra: the shortest path simulates, in each case, a specific vector between 2 points of an equilateral triangular grid on a flat surface (figure 2). On the icosahedron, the shortest path (which we will call the GPS for geodesic path segment) runs along the edge connecting the two marked vertices. This GPS and the two triangles touching it were lifted off the surface of the icosahedron and flattened above, indicating the vector. In the endo-pentakis dodecahedron, the GPS runs through the midpoints of the faces between the two marked vertices, which in the flat translates into a line segment at 30^o to the grid. In the endo-pentakis icosi-dodecahedron, the GPS runs along two edges, tracing the diagonal of the hexagon drawn. The vectors (2-D) in figure 2 can each be seen as representating a specific deltahedron (3-D) in the triangular grid.

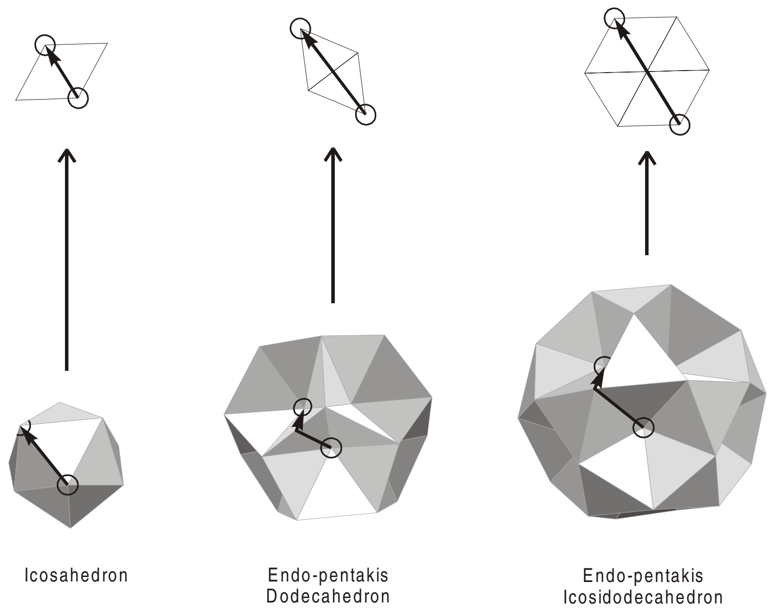

Figure 2: From deltahedra to vectors

From 2d vector to 3d deltahedron via a concentric net

Figure 3 shows all 5 deltahedra of figure 1 along with their corresponding vectors. In column A, the vector is depicted on part of the flat triangular grid together with a hexagon centered at each extremity.

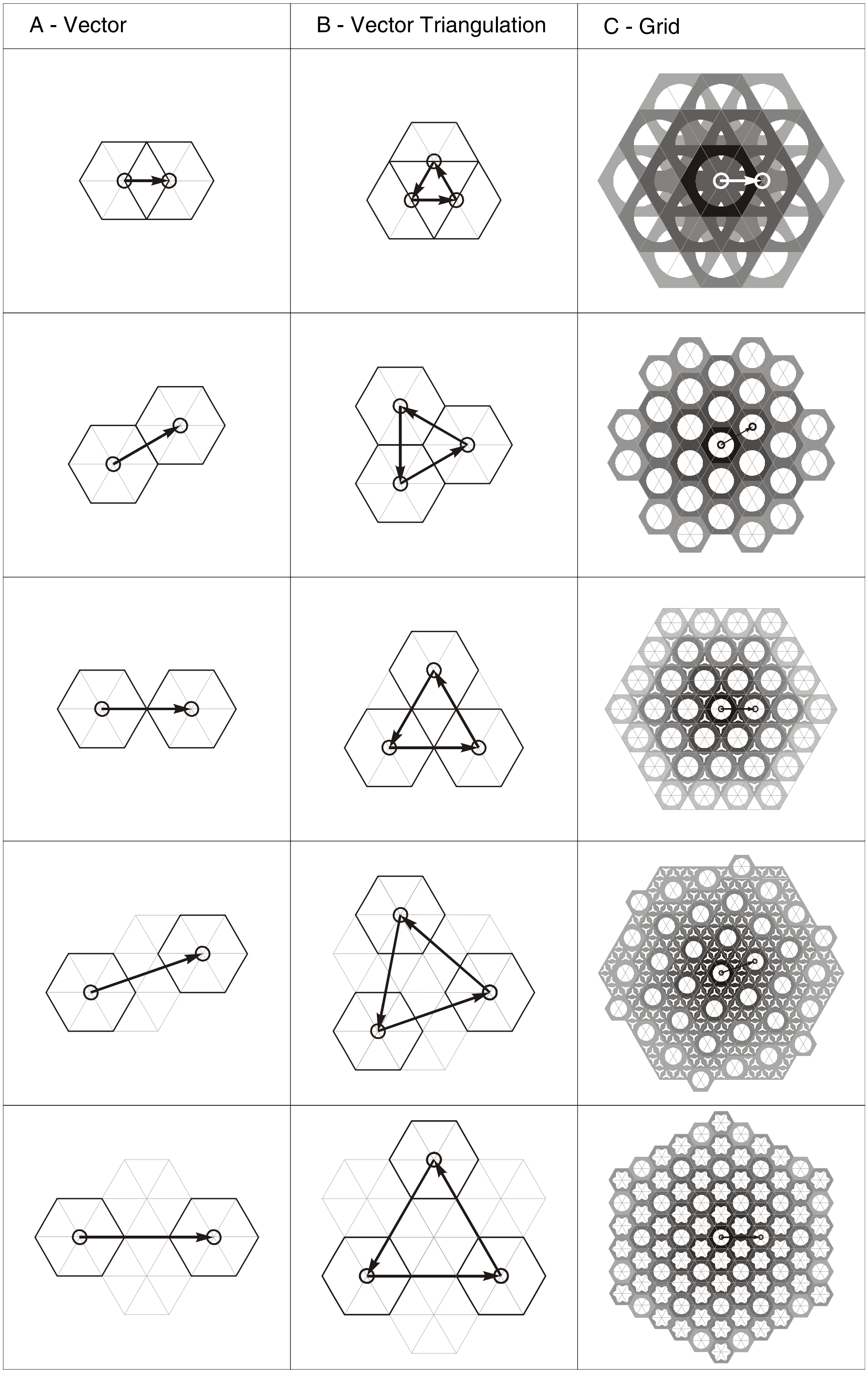

Figure 3: From vectors to deltahedra

- Simple circular motif on hexagon to become pentagonal
- Wavy motif on triangles between
- Wavy motif on hexagons between

Each of these hexagons will collapse, by losing a triangle, into a pentagonal pyramid. They will then become a 5 triangle-vertex on the deltahedron (see column E). Thus, these two hexagons yield a closest pair of 5-vertices in the deltahedron.

In column C, the same vector is embedded in the triangular grid.gUsing the 6-fold rotational symmetry of the grid we generate the vector system[2] that determines the relative positions of the grid vertices that can become 5-vertices. The rendering, based on the vector system, emphasizes the different elements in the resulting deltahedron. The hexagons with simple circular motif become pentagonal pyramids with a central 5-vertex. The triangles and hexagons with the wavy motif remain triangles and hexagons on the surface of the deltahedra. The grids on this column show the 6-fold rotational symmetry of the vector systems corresponding to each deltahedron.

Building the concentric net for the icosahedron (column D), we use the wedge technique [6: 133]. Starting at the centre of the grid and removing a 60^o wedge, we convert it into a 5-vertex when the net is closed. Moving out radially to the next ring of points to become 5-vertices (using the same vector system), we remove 5 new 60^o wedges and so on as shown until we have enough triangles to construct the deltahedron without overlaps. The last 5-vertex (antipodal to the first) only needs one triangle at the end of each of the 5 branches of the net (there are only 5 branches since the 5-fold symmetry of the deltahedronmeant the removal of one branch with the first wedge cut). The vertices where the wedges are removed correspond to the centers of the simple circular motif in column C. In the case of the icosahedron, the endo-pentakis-icosi-dodecahedron and the endo-pentakis trunkated icosahedron, the wedges start at each vertex of the net perimeter of column D, except for the last 5-vertex. The net is concentric because it has a central vertex around which the triangles are positioned in concentric rings corresponding to concentric rings on the deltahedron. The net gives a flattened view of the deltahedron as seen from the first 5-vertex.

It is worth noting here that the same net can produce different regular deltahedra. We could join up the net for the endo-pentakis-icosi-dodecahedron to make a large size icosahedron by having the 5-vertices point out instead of dimpling in. However, both shapes have the same number of triangles with the same edge connections. In one case the vertices point out and in another they point in. We can therefore say that a net gives a unique deltahedron, providing we assume the concavity of the 5-vertices where there is a choice[3] and we consider mirror images to be the same deltahedron[4].

Choosing one of six equivalent vectors

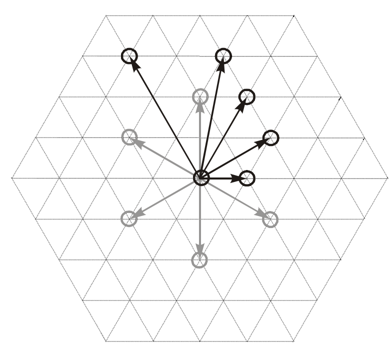

Figure 4: Representative vectors

[2] The vector system is defined as the vector, its images under 60^o rotation and combinations of these under vector addition.
[3] In the case of the icosahedron, the 5-vertices are necessarily convex because of their proximity.
[4] This applies to the endo-pentakis snub icosahedron, which we will discuss lower.

Figure 4 shows the representative vectors for the deltahedra of figure 1 on a triangular grid. Now that we have a method of constructing deltahedra from 2-D vectors we can take any vector on the grid and see what deltahedron it represents. When constructing the net, we use six rotated images of the vector. These are shown in gray in figure 4 for one of the vectors. We then see that each of the six rotated vector images represent the same deltahedron. Instead of drawing the whole grid and getting six copies of everything, we can simply draw one sixth of each set to get all the representative vectors without duplicates. In fact in figure 5 we have reduced the grid further to only a wedge of angle 30 degrees. This eliminates another kind of duplicate: mirror images of snub, or handed, deltahedra. The vector of a snub deltahedron if reflected in the horizontal base of the wedge will generate the mirror image of the snub deltahedron. Another way of seeing this duplication is: if the net in figure 3 column D is closed up with the facing surface on the outside yields a left handed snub deltahedron, then if it is closed up with the facing surface on the inside, the equivalent right handed snub deltahedron is built. Are there any more duplications? The answer here is no, because any two distinct vectors in the remaining wedge will either have different lengths or different sizes of angle from the horizontal in the grid, or both. Therefore according to this method, they will make different deltahedra.

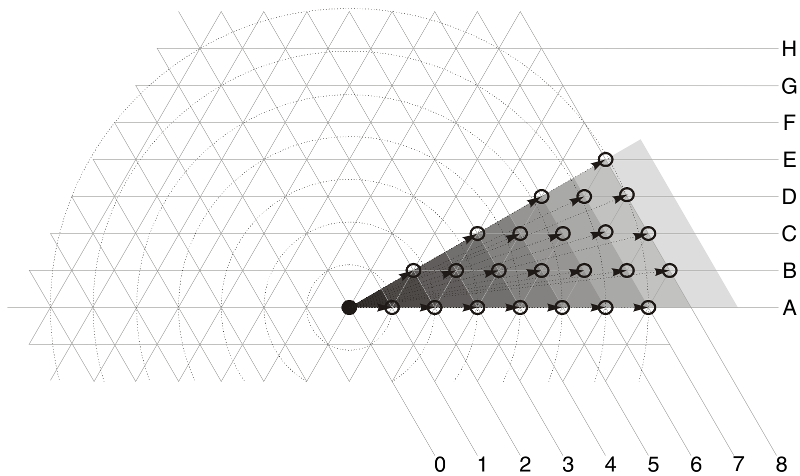

Figure 5: Wedge of vectors

From the wedge of vectors to the list of all regular deltahedra

We have now shown by construction that all regular deltahedra have a unique representative vector in our wedge. We have also shown that each vector can be used to construct a unique deltahedron. Therefore our vectors can act as a list of identifications for deltahedra. Using this identification, is it possible to create a systematic list of the regular deltahedra belonging to this family (as defined in the introduction)? Using the length of the vectors would be a great idea if it worked because simpler deltahedra having fewer triangles will have shorter vectors and come before more complex deltahedra in the list. This works only until the vector length equals 7 (the unit length corresponds to the edge length of a triangle). At a distance of 7 units from the origin, there are 2 points of the grid (7A and 8D in figure 5). Using instead the coordinate system of figure 5, we can use the rows and columns to list the vectors (and the deltahedra). In this particular case, because the rows are of infinite length, we will go 'column' by 'columns'.

In table 1, we list the vectors shown in the wedge of figure 5. To facilitate the use of the pythagorian theorem, we define n as follows: $n/2$ = the length of the horizontal component of the vector. This garantees that n is an integer since all points are horizontally at a multiple of ½ from the origin. Similarly, we define *m* as follows: $m(\sqrt{3})/2$ = the length of the vertical component of the vector. The third column lists the square value of the vector length, and in the last column, we have shown the names of the known deltahedra corresponding to their vectors in the table.

Vector	**n**	**m**	L^2	**Deltahedron**
1-A	2	0	1	Icosahedron
2-B	3	1	3	E-P Dodecahed.
2-A	4	0	4	E-P Icosidodecah.
3-B	5	1	7	E-P Snub Icosah.
3-A	6	0	9	E-P Truncated Icosah.
4-C	6	2	12	...
4-B	7	1	13	...
4-A	8	0	16	...
5-C	8	2	19	...
5-B	9	1	21	...
5-A	10	0	25	...

Vector	**n**	**m**	L^2	**Deltahedron**
6-D	9	3	27	...
6-C	10	2	28	...
6-B	11	1	31	...
6-A	12	0	36	...
7-D	11	3	37	...
7-C	12	2	39	...
7-B	13	1	43	...
7-A	14	0	**49**	...
8-E	12	4	48	...
8-D	13	3	**49**	...
8-C	14	2	52	...

Table 1: Squared distance table for Vectors and their Deltahedra

According to pythagorus, the squared length L^2 of a vector is given by:

$$L^2 = \left(\frac{n}{2}\right)^2 + \left(\frac{\sqrt{3}\times m}{2}\right)^2,$$

where *n* and *m* are positive integers.

In the example of figure 6 (vector 2-B), $n = 3$ and $m = 1$, giving a vector length of 3:

$$L^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}\times 1}{2}\right)^2 = \left(\frac{9}{4}\right) + \left(\frac{3}{4}\right) = \frac{12}{4} = 3,$$

Notice that vectors 7-A and 8-D are both of squared length 49 as shown in figure 5 and discussed above. We also see in the table that L^2 always seems to be an integer and that *n* and *m* always seem to be both odd or both even.

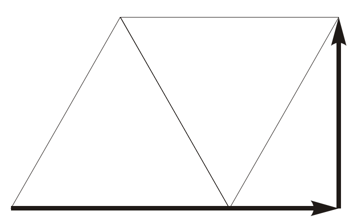

Figure 6: Vector 1B

The relative parity of N & M determines the integral value of L^2

Some interesting numerical properties of this system can be proven. First, the geometry of the wedge determines that if n is even, then so is m and conversely, if *n* is odd, so is *m*[5]. Symbolically,

[5] If we study the grid points in the wedge of figure 5, by rows, the pattern is clear: even 'numbered' rows have even values for n and m and odd 'numbered' rows have odd values for *n* & *m*.

$\frac{(n+m)}{2} = p$, a positive integer, where $p > m, n$.

Substituting for *n*, we get:

$$n = 2p - m,$$

and in the Pythagoras equation,

$$L^2 = \left(\frac{2p-m}{2}\right)^2 + \left(\frac{\sqrt{3}\times m}{2}\right)^2$$

$$= \frac{4p^2}{4} - \frac{4mp}{4} + \frac{m^2}{4} + \frac{3m^2}{4}$$

$$= p^2 - mp + m^2$$

where *m* and *p* are positive integers and because $p > m$, we know that the last line is positive.

Relating the length of the vector to the surface area of the deltahedron

Returning to the cells of column B, figure 3, we take a closer look at a simple vector triangle from the vector system. The area of this vector triangle is calculated from the length of the vector as being L^2. As we have seen previously, that value is always an integer. Illustrating this using figure 3, in the first row the vector is of unit length and the triangle in column B is a single triangle (1 tile) of the grid. For the vector of length 2, as in the third row, the triangle is made up of 4 grid triangles. Therefore, the vector of squared length $2^2 = 4$ defines a vector triangle of area 4. From all this, we can make the following statement:

The vector triangle defined in each vector system shown in this paper has a surface area equal to the square of the length of the vector. Furthermore, this value is always an integer.

Comparing the value of L^2 to the number of triangles on the surface of the equivalent deltahedron (see column F) we observe that the total number of triangles is always 20 times the squared vector length and the area of the vector triangle, L^2. Does that mean that it always takes 20 vector triangles to make the whole polyhedron? We can see this is true in the case of rows 1, 3 and 5 of figure 3 (see column D). The 3 nets only differ in the relative scale of the grid and the outline. In fact, comparing the area of a single triangle in the net of row 1, the equivalent section of the net in row 3 has an area of 4 (L^2, where L=2), and the equivalent section of the net in row 5 has an area of 9 (L^2, where L=3). In the other rows, the vector triangle of column B allow us to deduce the same relationship (in row 2, the area is 3 and in row 4, 7). If we draw the vector systems directly on the deltahedra, with a vector connecting each neighbouring pair of 5-vertices as we started in the lower part of figure 2, we see that there are indeed always 20 vector triangles on each deltahedron of this class. This shows that the surface area is always $20\times L^2$, where L is the vector length, and L^2 is an integer. In other words,

The deltahedra of this class are always composed of an integer multiple of 20 triangles. That multiple is in fact the squared length of the defining vector.

Note that the minimum number of 20 faces is achieved by the icosahedron.

Conclusion

In the course of this paper, we have shown that the classical operations of stellation, truncation and subdivision do not enable us to generate all the deltahedra of a specific class. More accurately, asymmetric or handed deltahedra cannot be generated from a platonic solid by these processes. Examining the relationship between nearest 5-vertices on our regular deltahedra allowed us to develop a 'vector system' method that can be used to list and generate all deltahedra of a class. We then applied this method to the class specified in the introduction.

Later, by examining the length properties of admissible grid vectors we were able to prove that for all deltahedra with the specified regularity property the number of faces is divisible by 20.

We have not however proven that every admissible grid vector gives rise to a buildable deltahedron. We do not know if all the triangles can always fit together in three-dimensional space with no gaps or overlaps. It is therefore possible that there may be some impossible deltahedra in our list. Evidence collected to this point seems to support the assumption that there are no such impossible deltahedra and we are optimistic that a proof can be produced by considering the way in which the vector triangles from figure 3, column B can be made to fit together in three space.

Recent developments in organic chemistry [1] have led to carbon molecules that take on shapes such as the truncated icosahedron, the soccer ball. The geometric building blocks of these molecules include regular hexagons and pentagons formed by rings of carbon atoms with fixed bond length. The equivalent deltahedron would have six flat triangles making a hexagon, and five triangles coming together to form a pyramid with a pentagonal base. There is therefore a precise geometric correspondence between carbon molecules and deltahedra. This is enough to provide additional motivation for studying the geometry of deltahedra, the polyhedra with equilateral triangular faces.

A further exercise for the reader would be to consider how to apply the work in this paper to the class of deltahedra with 4-vertices and 6-vertices and the octahedral symmetry group. Do we get the result that there are always an integer multiple of 8 faces in that class of deltahedra?

A different kind of situation arises if we combine 6-vertices and 7-vertices. The result is a hyperbolic saddle surface which never closes up and has an infinite number of faces.

Finally, future work could consider the classification of families of deltahedra with certain symmetry groups for given combinations of vertices. One such examples is the class of deltahedra with 3-vertices, 6-vertices and 8-vertices (which includes the stella-octangula possessing the octahedral symmetry group and 24 faces). Does every deltahedron in one of these families have an integer multiple of a number of faces, and does there exist a deltahedron that has the minimum possible number of faces?

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